Coded many-user multiple access via Approximate Message Passing

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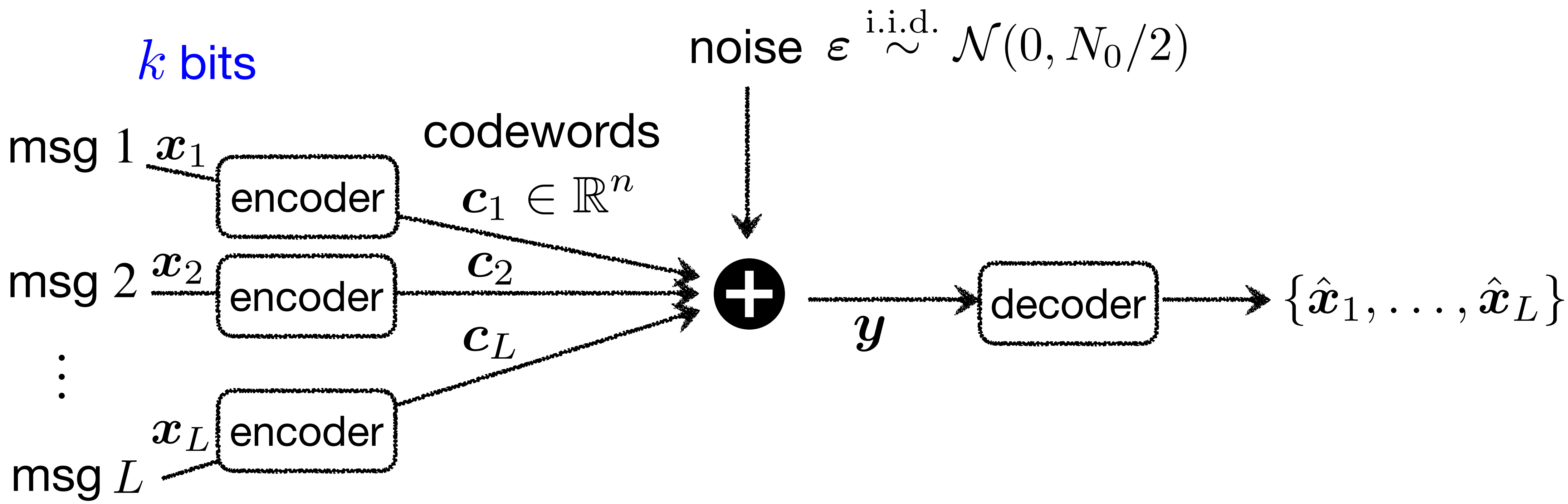
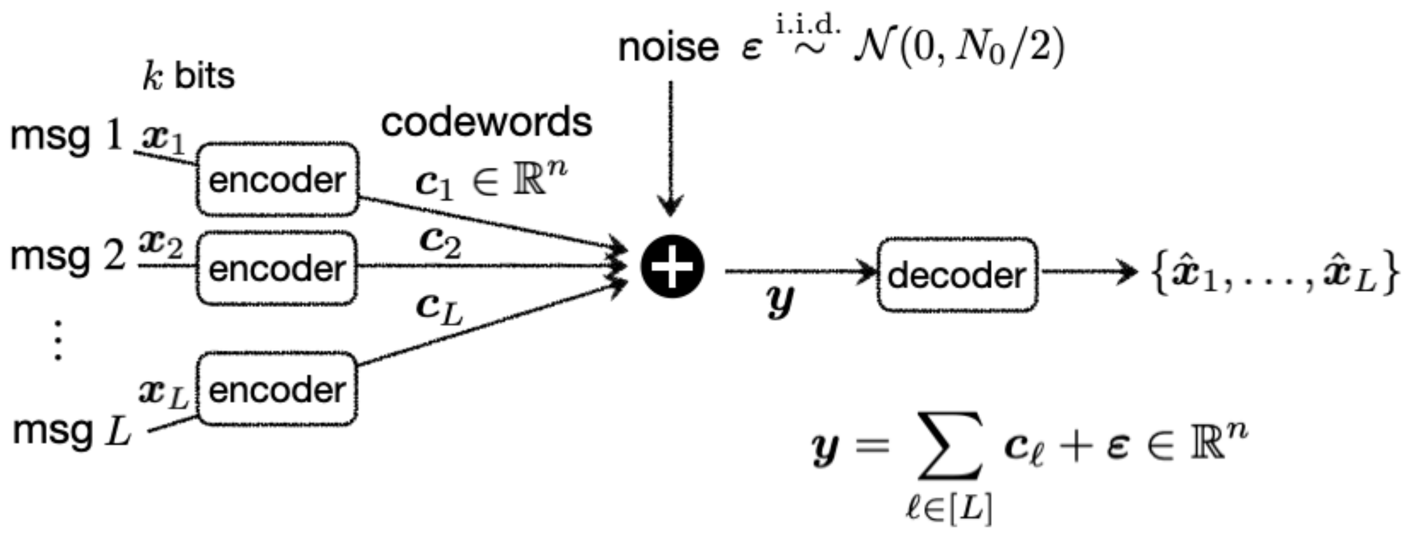
Gaussian multiple-access channel (GMAC)



$$\boldsymbol{y} = \sum_{\ell \in [L]} \boldsymbol{c}_\ell + \boldsymbol{\varepsilon} \in \mathbb{R}^n$$



Many-user setting

- user density $\mu := L/n$
- fixed user payload k bits
- energy-per-bit constraint: $\|\boldsymbol{c}_\ell\|_2^2 \leq E := E_b k$
- users have distinct codebooks

Linear scaling regime: $L, n \to \infty$ with μ fixed.

Given μ , what is minimum E_b/N_0 required to achieve a target error rate?

e.g. $P_e = \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{P}(\hat{\boldsymbol{x}}_\ell \neq \boldsymbol{x}_\ell)$

[Chen, Chen, Guo, '17], [Ravi, Koch '19], [Polyanskiy '17], [Zadik, Polyanskiy, Thrampoulidis '19], [Polyanskiy, Kowshik '20]

Previous work [Polyanskiy '17], [Zadik, Polyanskiy, Thrampoulidis '19], [Polyanskiy, Kowshik '20]

What can be achieved without memory or computational constraints?

random Gaussian codebooks + maximum-likelihood decoding

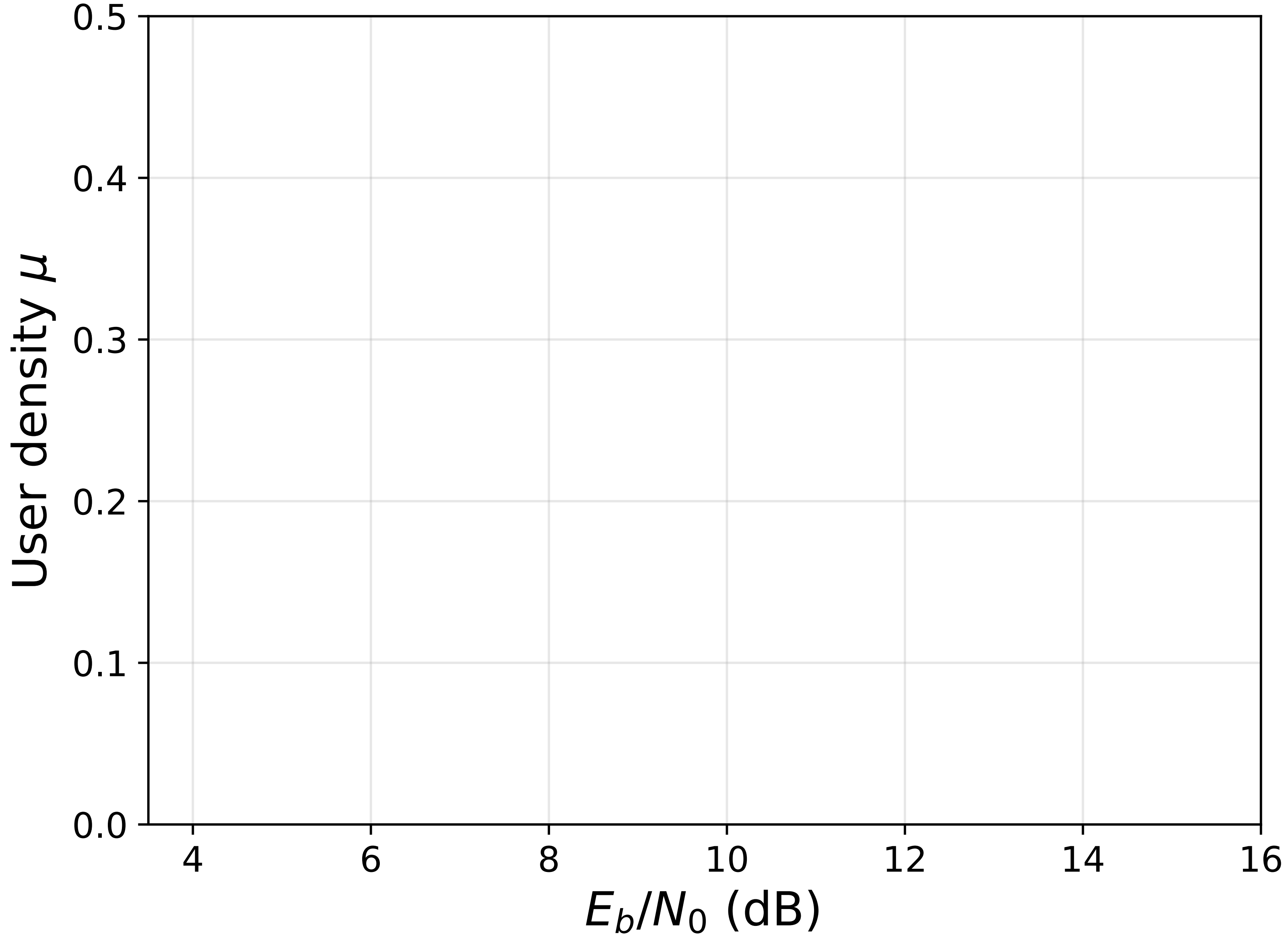
This talk [Hsieh, Rush, Venkataramanan '22], [Liu, Hsieh, Venkataramanan '24]

What can be achieved with efficient coding schemes?

random linear coding + Approximate Message Passing (AMP) decoding

Small user payload $k = 8$ bits,

target $P_e \leq 10^{-4}$



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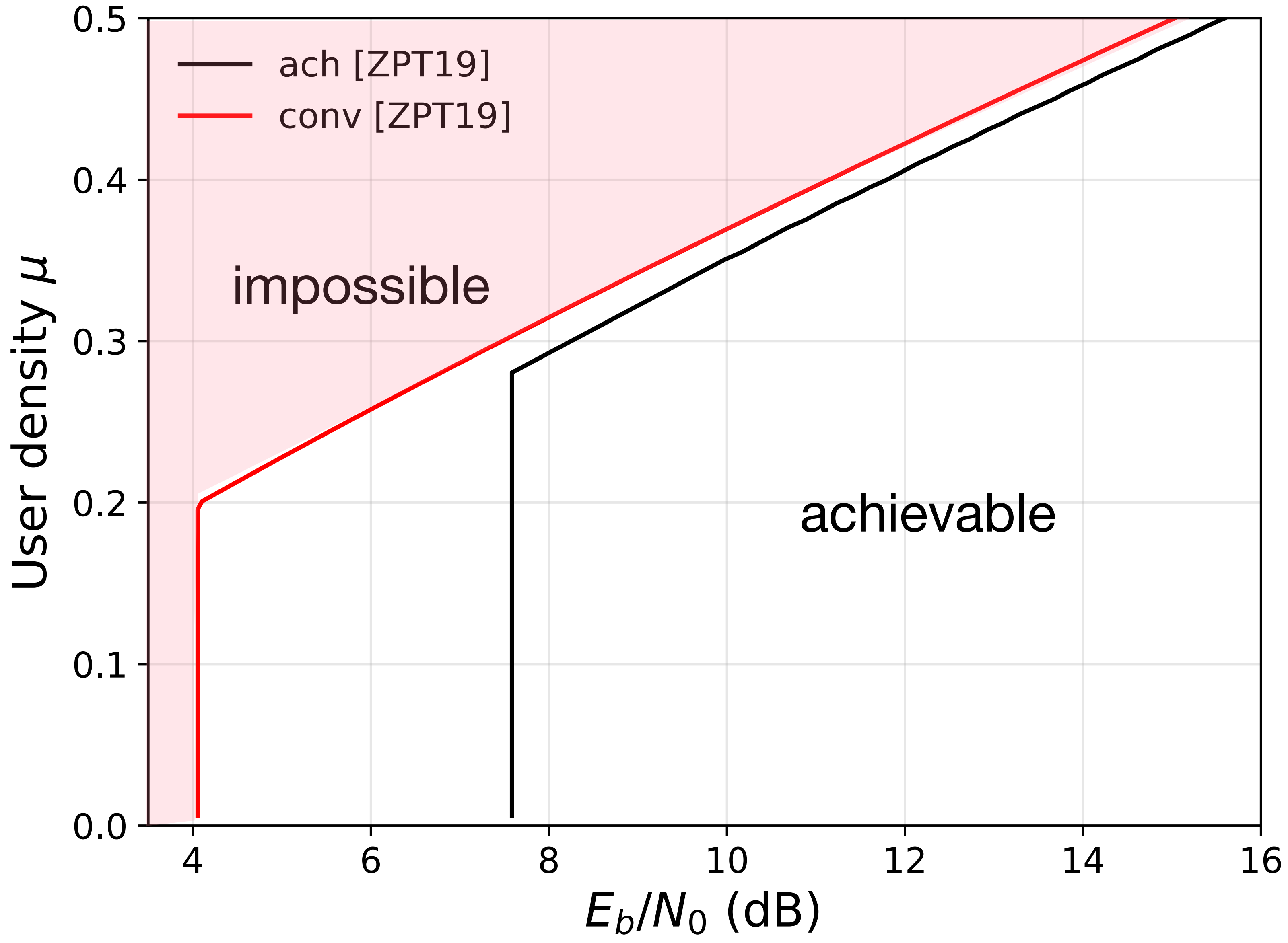

[Zadik, Polyanskiy, Thrampoulidis '19] random Gaussian codebooks + ML decoding

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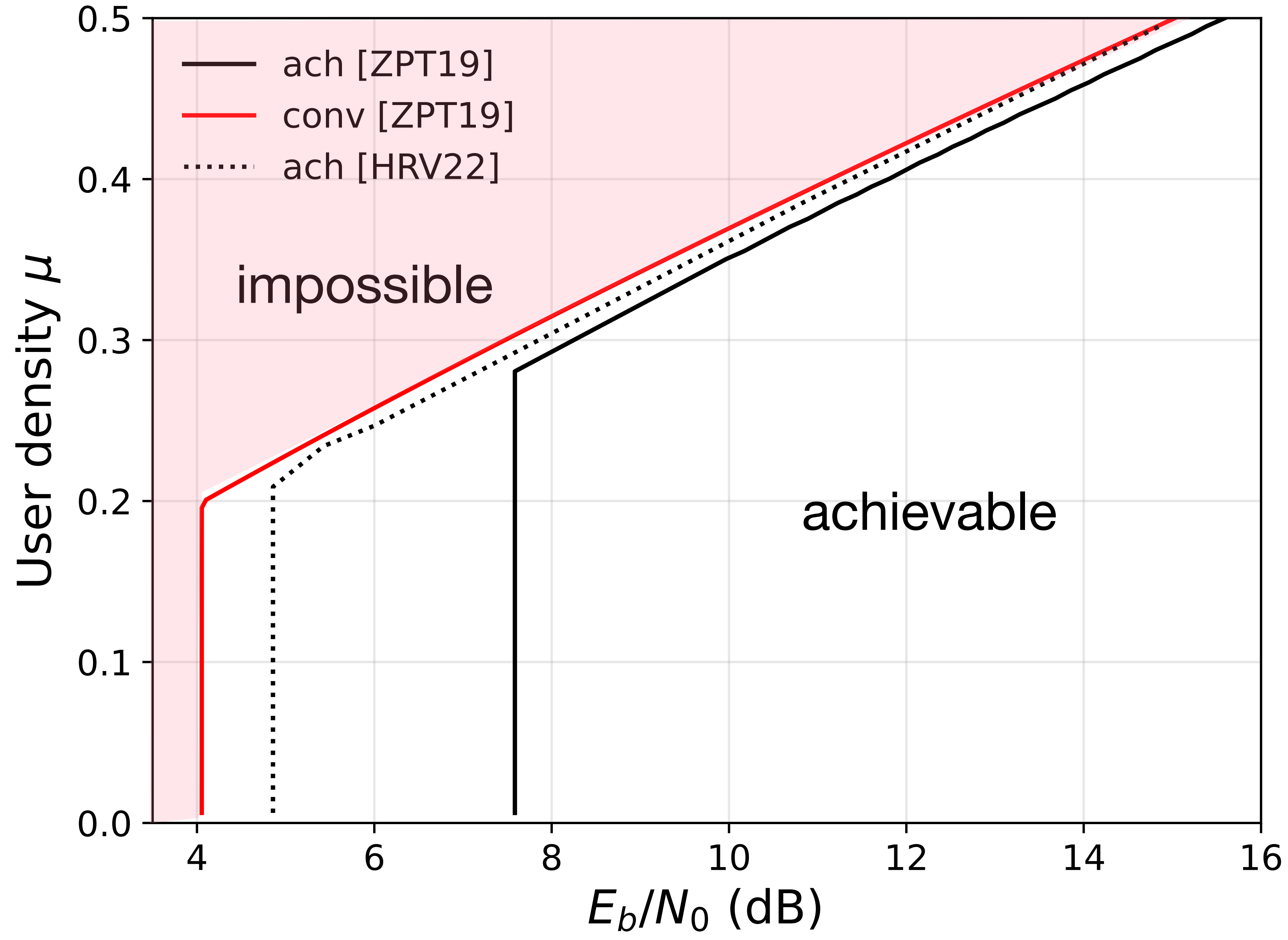
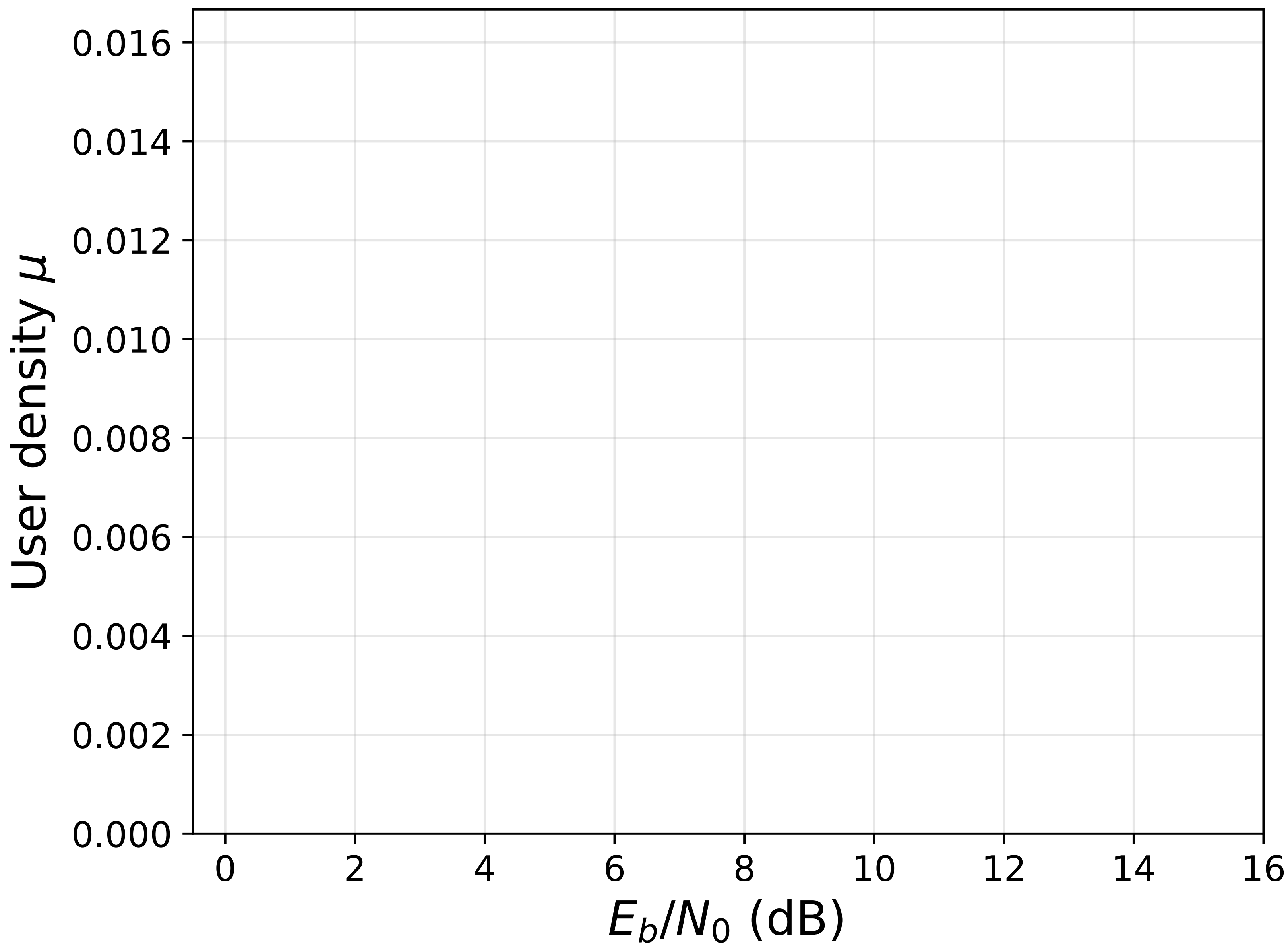


[Zadik, Polyanskiy, Thrampoulidis '19] random Gaussian codebooks + ML decoding

[Hsieh, Rush, Venkataramanan '22] sparse linear regression codes + AMP decoding (memory and computational costs both $\propto 2^k$)

Large user payload $k = 240$ bits,

target $P_e \leq 10^{-4}$



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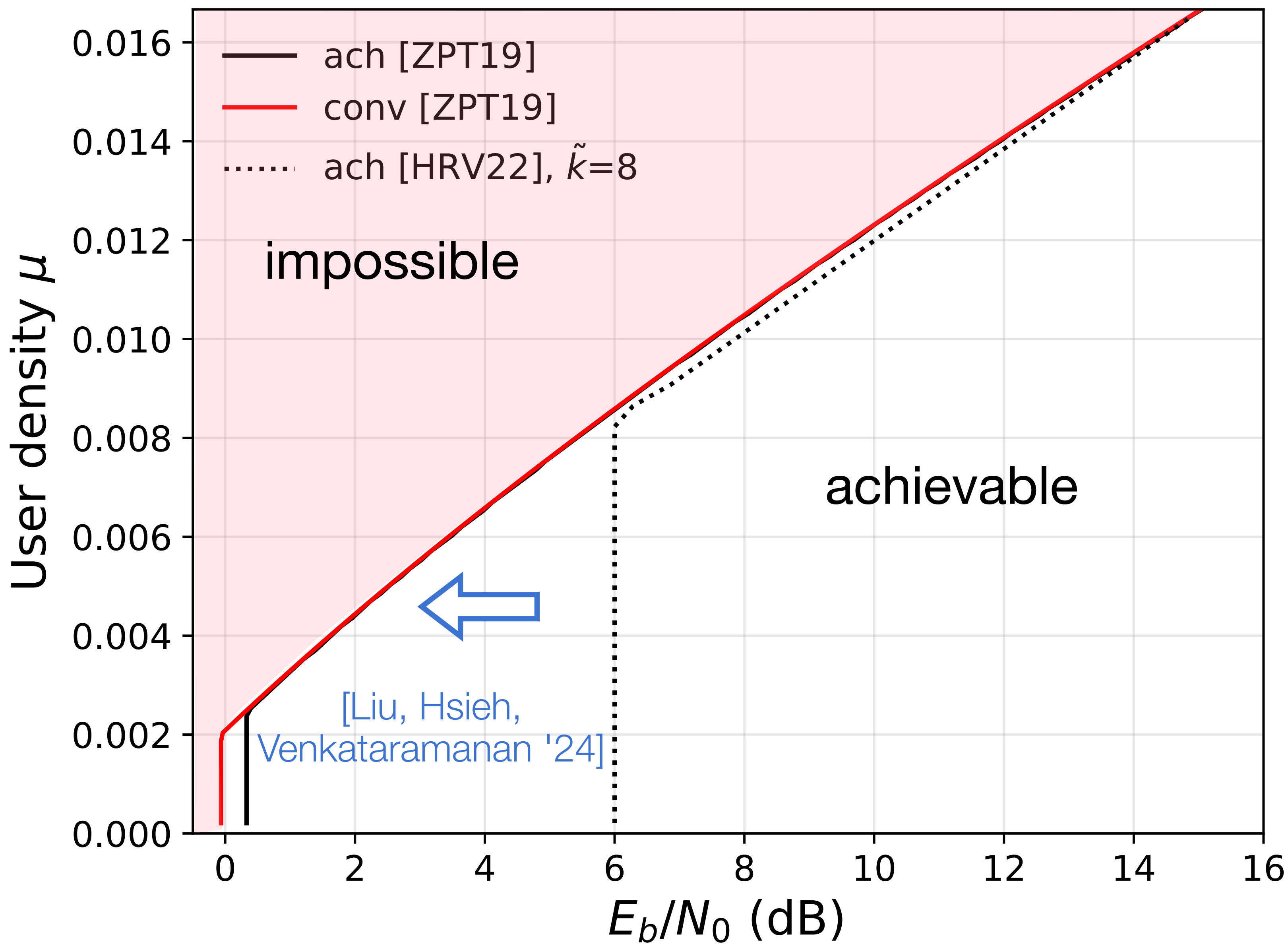
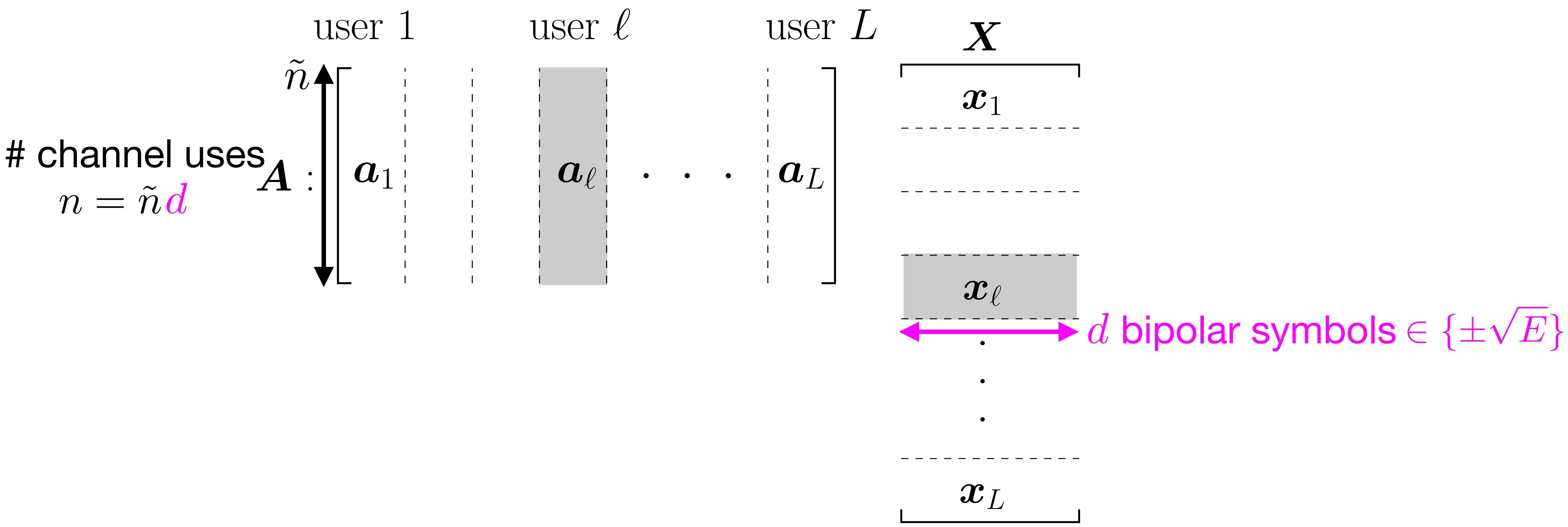


[Zadik, Polyanskiy, Thrampoulidis '19] random Gaussian codebooks + ML decoding

[Hsieh, Rush, Venkataramanan '22] sparse linear regression codes + AMP decoding

Random binary-CDMA + outer code



For each user $\ell \in [L]$:

- Random signature sequence: $\boldsymbol{a}_\ell \in \mathbb{R}^{\tilde{n}}$
- Outer (k, d) linear code: k bits payload encoded in $\boldsymbol{x}_\ell \in \{\pm\sqrt{E}\}^d \sim p_{\bar{\boldsymbol{x}}}$
  e.g. LDPC

Channel output: $\boldsymbol{Y} = \sum_{\ell \in [L]} \boldsymbol{a}_\ell \boldsymbol{x}_\ell + \boldsymbol{\mathcal{E}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{\mathcal{E}} \in \mathbb{R}^{\tilde{n} \times d}$

memory and computational costs linear in k

AMP decoder for i.i.d. Gaussian A

Given $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{\mathcal{E}}$ and $\boldsymbol{A}$, recover $\boldsymbol{X}$

Start with initialiser $\boldsymbol{X}^0 = \boldsymbol{0}$, for $t \geq 1$ debias term

$$\boldsymbol{Z}^t = \boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}^t + \frac{1}{\tilde{n}}\boldsymbol{Z}^{t-1}\Big[\sum_{\ell\in[L]} \eta'_{t-1}\left(\boldsymbol{s}_\ell^{t-1}\right)\Big]^\top$$

$$\boldsymbol{X}^{t+1} = \eta_t(\boldsymbol{S}^t), \quad \boldsymbol{S}^t = \boldsymbol{A}^\top \boldsymbol{Z}^t + \boldsymbol{X}^t$$

End with hard-decision estimate $\hat{\boldsymbol{X}}^{t+1} = h_t(\boldsymbol{S}^t)$

- $\eta_t : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz and applies **row-wise** to matrix inputs
- debias term ensures empirical dist. of rows of $(\boldsymbol{S}^t - \boldsymbol{X}) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}^t)$
- η_t estimates $\boldsymbol{X}$ from observation in Gaussian noise

can be deterministically computed

Theorem [Liu, Hsieh, Venkataramanan '24]

For $\boldsymbol{A} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1/\tilde{n})$, let $\eta_1, \ldots, \eta_t$ be Lipschitz, then

Asymp. user error rate (UER)

$$\lim_{L\to\infty} \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{1}\{\hat{\boldsymbol{x}}_\ell^{t+1} \neq \boldsymbol{x}_\ell\}$$

Asymp. bit error rate (BER)

$$\lim_{L\to\infty} \frac{1}{Ld} \sum_{\ell=1}^{L} \sum_{i=1}^{d} \mathbb{1}\left\{\hat{x}_{\ell,i}^{t+1} \neq x_{\ell,i}\right\}$$

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deterministic quantities

Asymp. bit error rate (BER)

$$\lim_{L\to\infty} \frac{1}{Ld} \sum_{\ell=1}^{L} \sum_{i=1}^{d} \mathbb{1}\left\{\hat{x}_{\ell,i}^{t+1} \neq x_{\ell,i}\right\} = \frac{1}{d} \sum_{i=1}^{d} \mathbb{P}\left(\left[h_t\left(\bar{x} + g^t\right)\right]_i \neq \bar{x}_i\right).$$

- $\bar{\boldsymbol{x}} \in \{\pm\sqrt{E}\}^d$ uniformly distributed among 2^k codewords
- $\boldsymbol{g}^t \in \mathbb{R}^d \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}^t)$ independent of $\bar{\boldsymbol{x}}$

How to choose denoiser $\eta_t : \mathbb{R}^d \to \mathbb{R}^d$?

computational cost

- **Bayes-optimal**

$$\boldsymbol{x}_\ell^{t+1} = \eta_t(\boldsymbol{s}_\ell^t) = \mathbb{E}\left[\bar{\boldsymbol{x}} \mid \bar{\boldsymbol{x}} + \boldsymbol{g}^t = \boldsymbol{s}_\ell^t\right]$$

$$= \sum_{\boldsymbol{x}'} \boldsymbol{x}' \cdot \frac{\exp\left(-\frac{1}{2}(\boldsymbol{x}' - 2\boldsymbol{s}_\ell^t)^\top (\boldsymbol{\Sigma}^t)^{-1} \boldsymbol{x}'\right)}{\sum_{\tilde{\boldsymbol{x}}'} \exp\left(-\frac{1}{2}(\tilde{\boldsymbol{x}}' - 2\boldsymbol{s}_\ell^t)^\top (\boldsymbol{\Sigma}^t)^{-1} \tilde{\boldsymbol{x}}'\right)}$$

$O(2^k d^3)$

- **marginal-MMSE**

$$\boldsymbol{x}_\ell^{t+1} = \eta_t(\boldsymbol{s}_\ell^t) = \begin{bmatrix} \mathbb{E}[\bar{x}_1 \mid \bar{x}_1 + g_1^t = s_{\ell,1}^t] \\ \vdots \\ \mathbb{E}[\bar{x}_d \mid \bar{x}_d + g_d^t = s_{\ell,d}^t] \end{bmatrix}$$

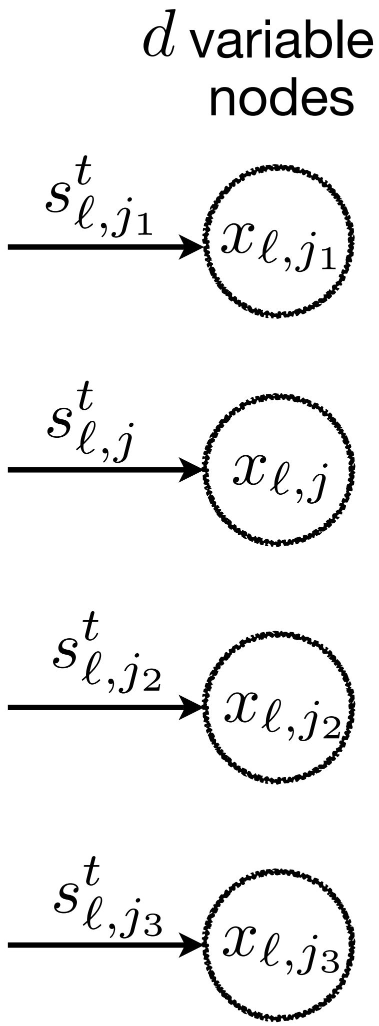
$O(d)$

- **Belief Propagation (BP)** [Amalladine et al. '22], [Ebert et al. '23]

$$\boldsymbol{x}_\ell^{t+1} = \eta_t(\boldsymbol{s}_\ell^t) \approx \begin{bmatrix} \mathbb{E}[\bar{x}_1 \mid \bar{x}_1 + g_1^t = s_{\ell,1}^t \text{ , parities involving } \bar{x}_1 \text{ satisfied}] \\ \vdots \\ \mathbb{E}[\bar{x}_d \mid \bar{x}_d + g_d^t = s_{\ell,d}^t \text{ , parities involving } \bar{x}_d \text{ satisfied}] \end{bmatrix}$$

$O(d)$

Belief Propagation (BP) denoiser $\boldsymbol{x}_\ell^{t+1} = \eta_t(\boldsymbol{s}_\ell^t)$



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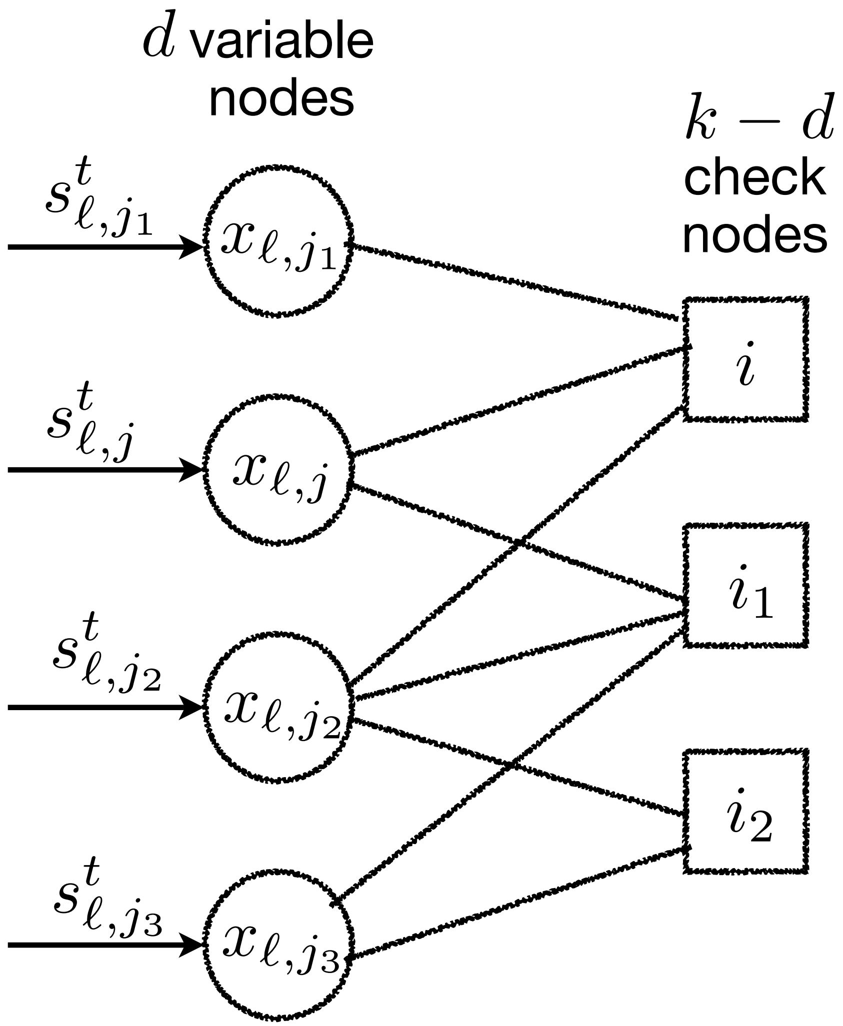

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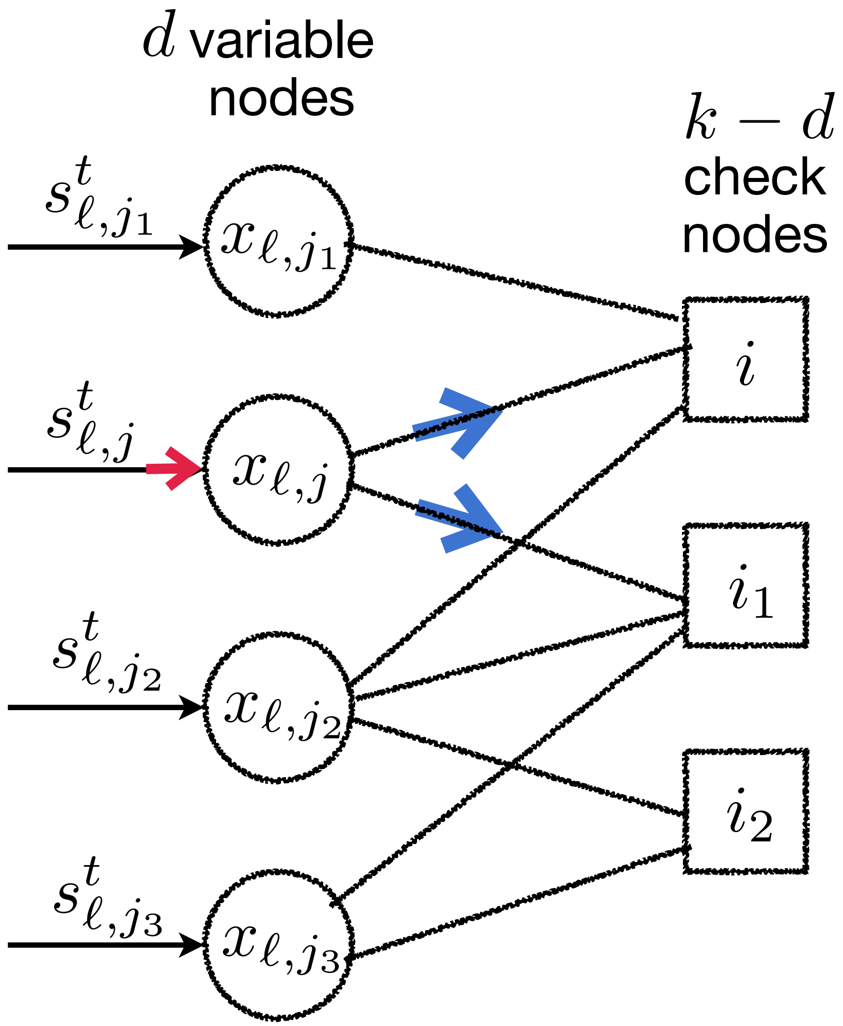

1. Initialise:

$$L^{(0)}_{j\to i} = \ln\left[\frac{p(s^t_{\ell,j}\,|x_{\ell,j} = +\sqrt{E})}{p(s^t_{\ell,j}\,|x_{\ell,j} = -\sqrt{E})}\right] = \frac{2\sqrt{E}s^t_{\ell,j}}{\Sigma^t_{j,j}} =: L(s^t_{\ell,j})$$

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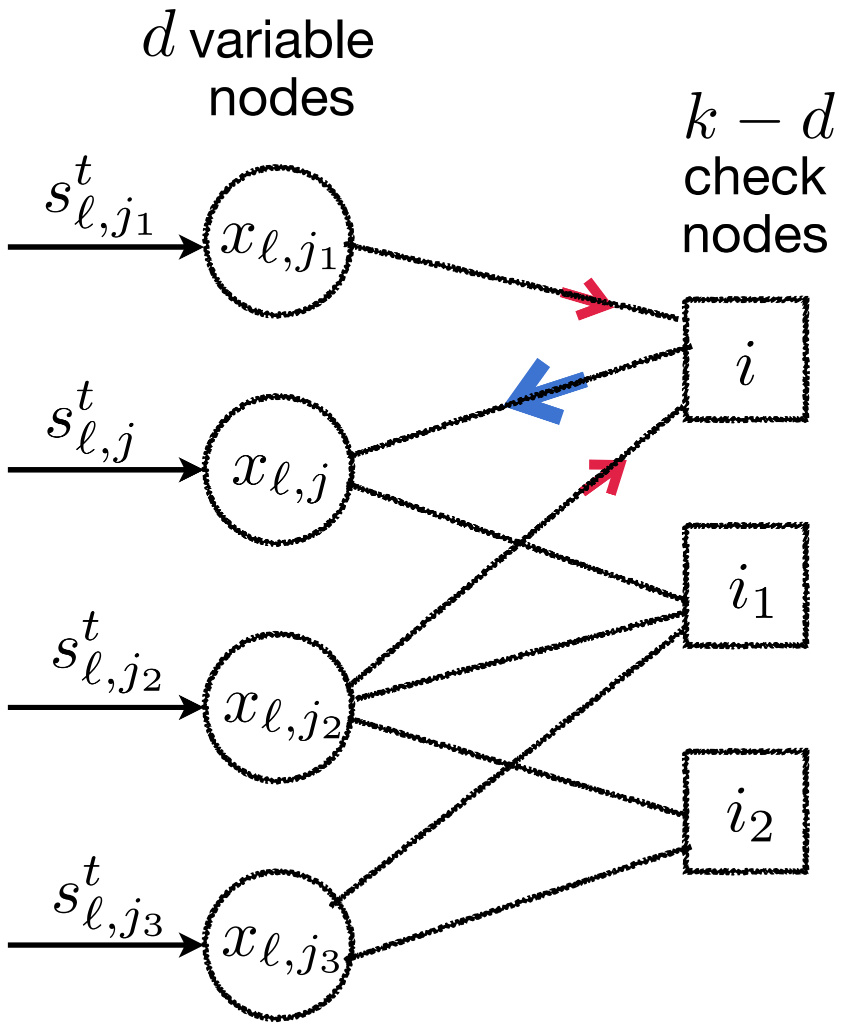

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2. Middle rounds:

$$L^{(r)}_{i\to j} = 2\tanh^{-1}\left[\prod_{j'\in N(i)\setminus j}\tanh\left(\frac{1}{2}L^{(r-1)}_{j'\to i}\right)\right]$$

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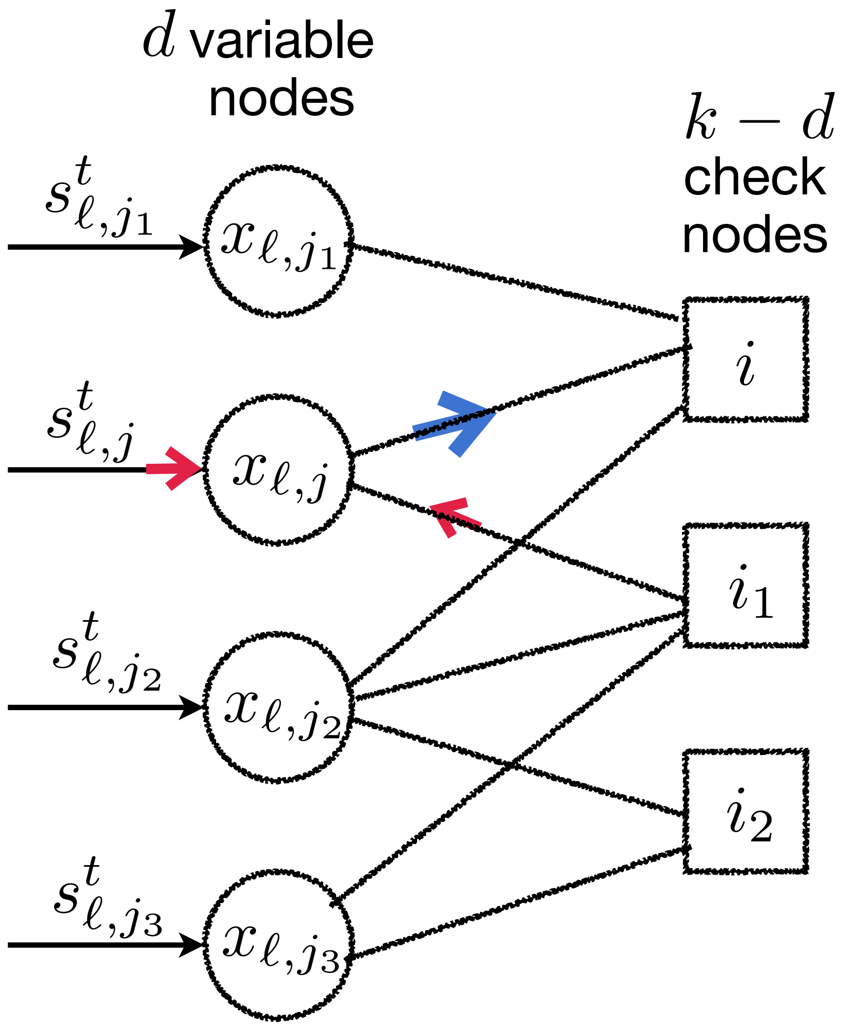

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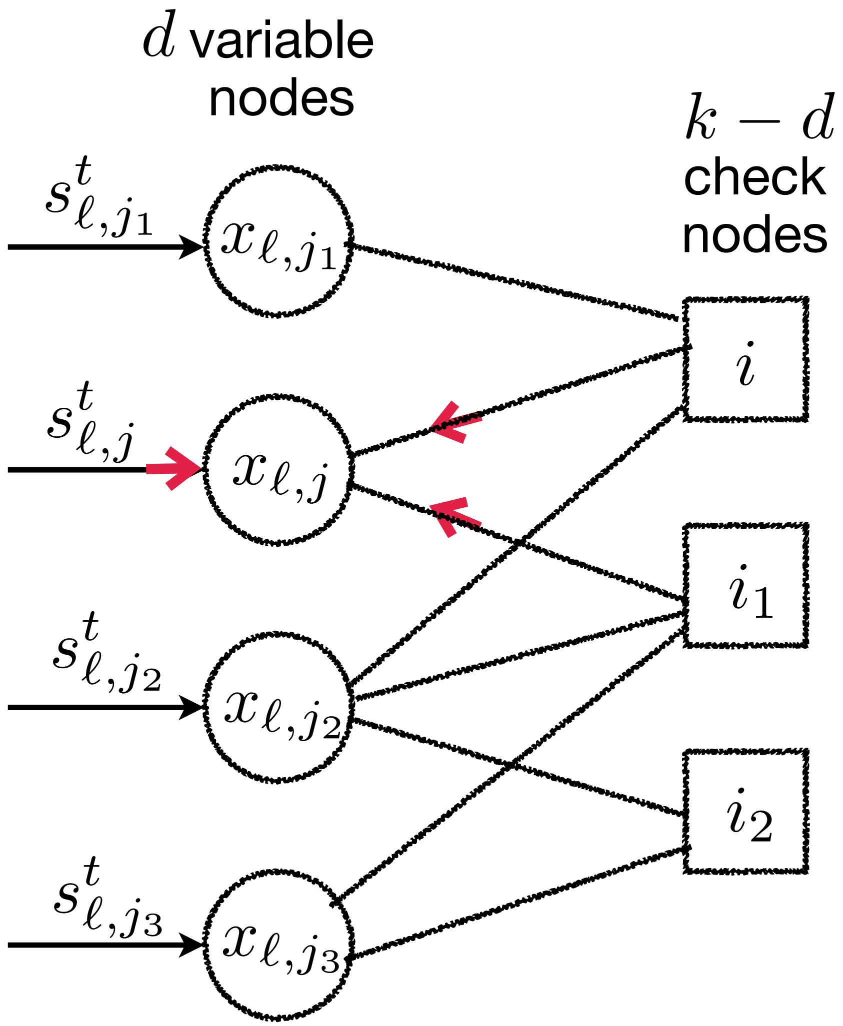

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approx. marginal posterior probabilities

4. Update AMP estimate:

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$$\approx \mathbb{E}\left[\bar{x}_j \mid \bar{x}_j + g^t_j = s^t_{\ell,j}\,,\ \text{parities satisfied}\right]$$

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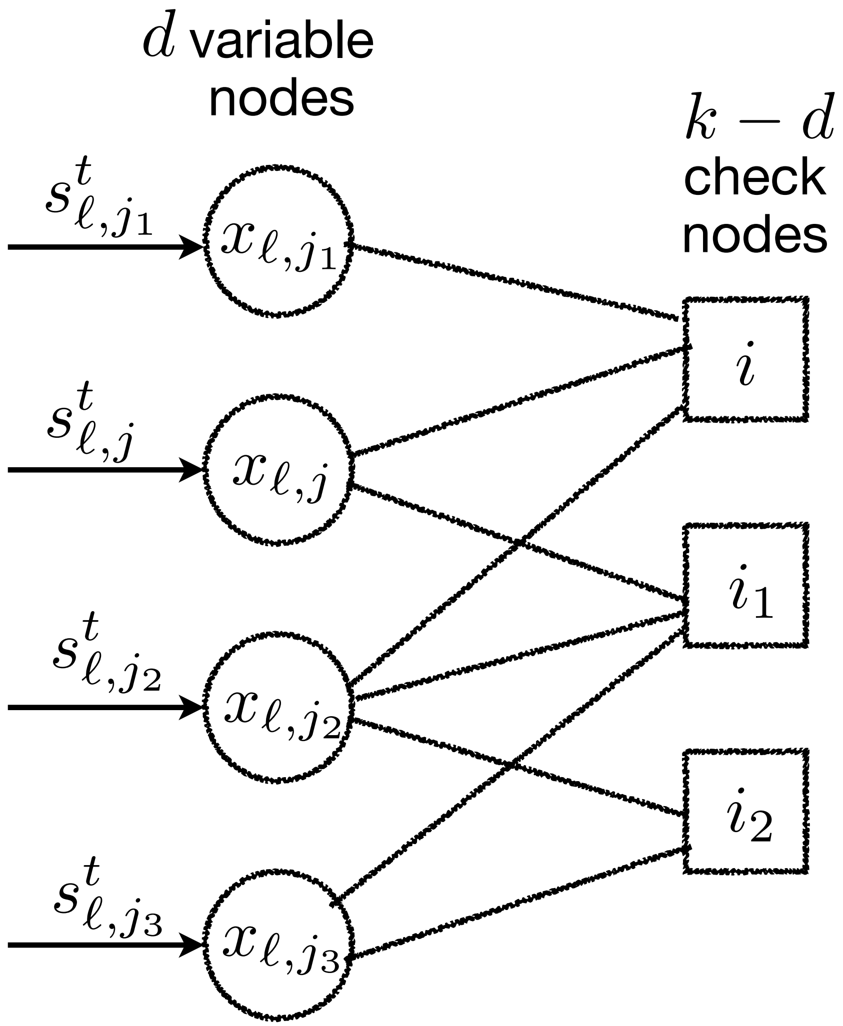

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Jacobian η'_t has closed form expression when $\mathcal{R}$ < girth of bipartite graph!

[Amalladine et al. '22], [Ebert et al. '23]

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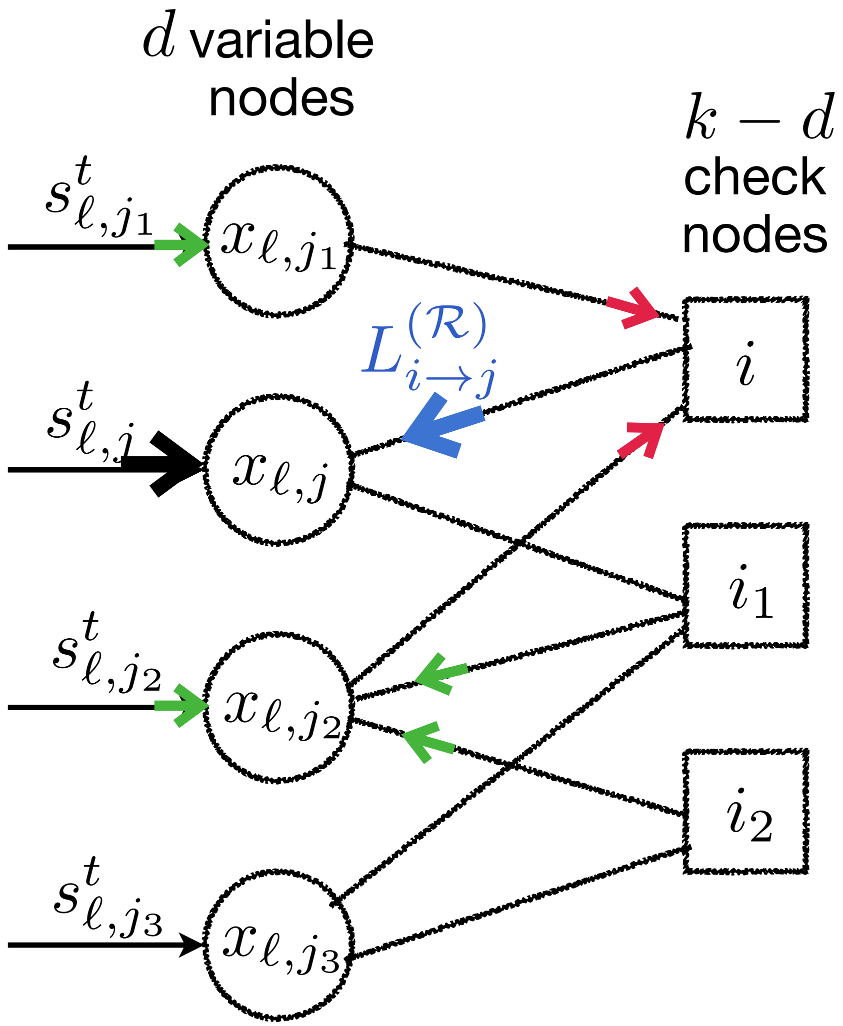

Jacobian η_t' has closed form expression
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Theorem

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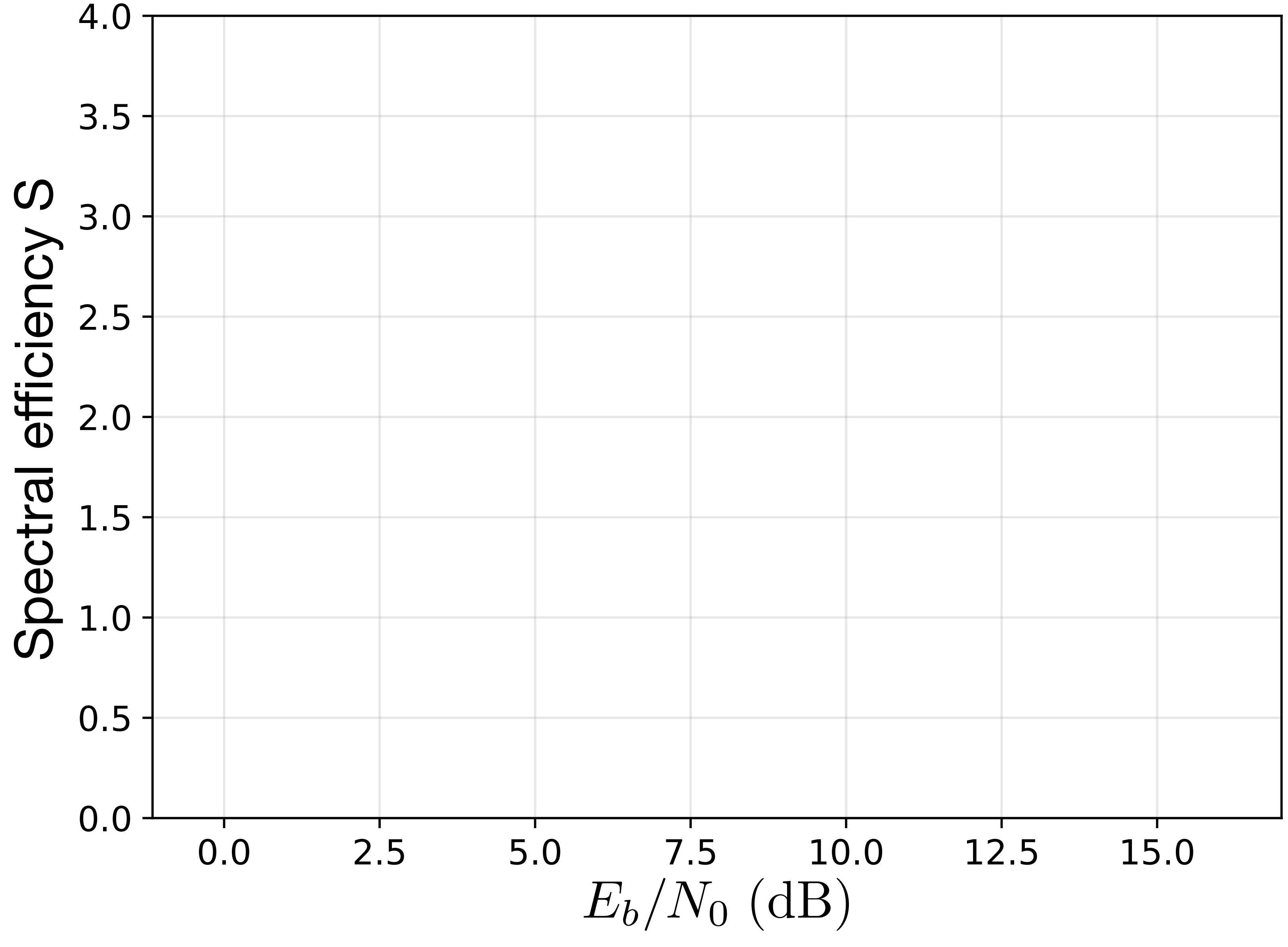
Applies to AMP with BP denoiser!

payload 360 bits, target $\texttt{BER} \leq 10^{-4}$



Spectral efficiency: total # bits / total # channel uses

$$\mathsf{S} = \mu k = Lk/n$$

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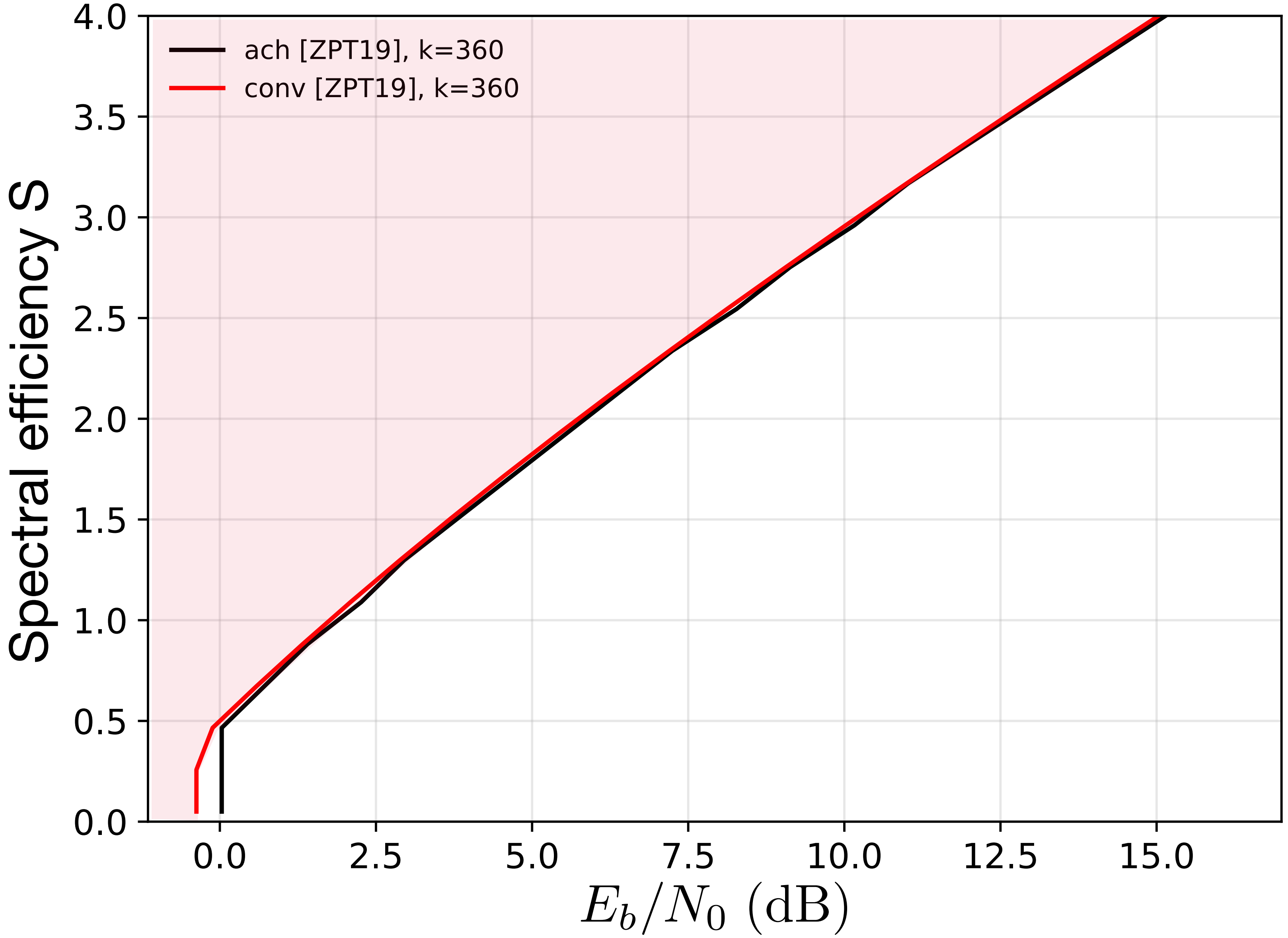

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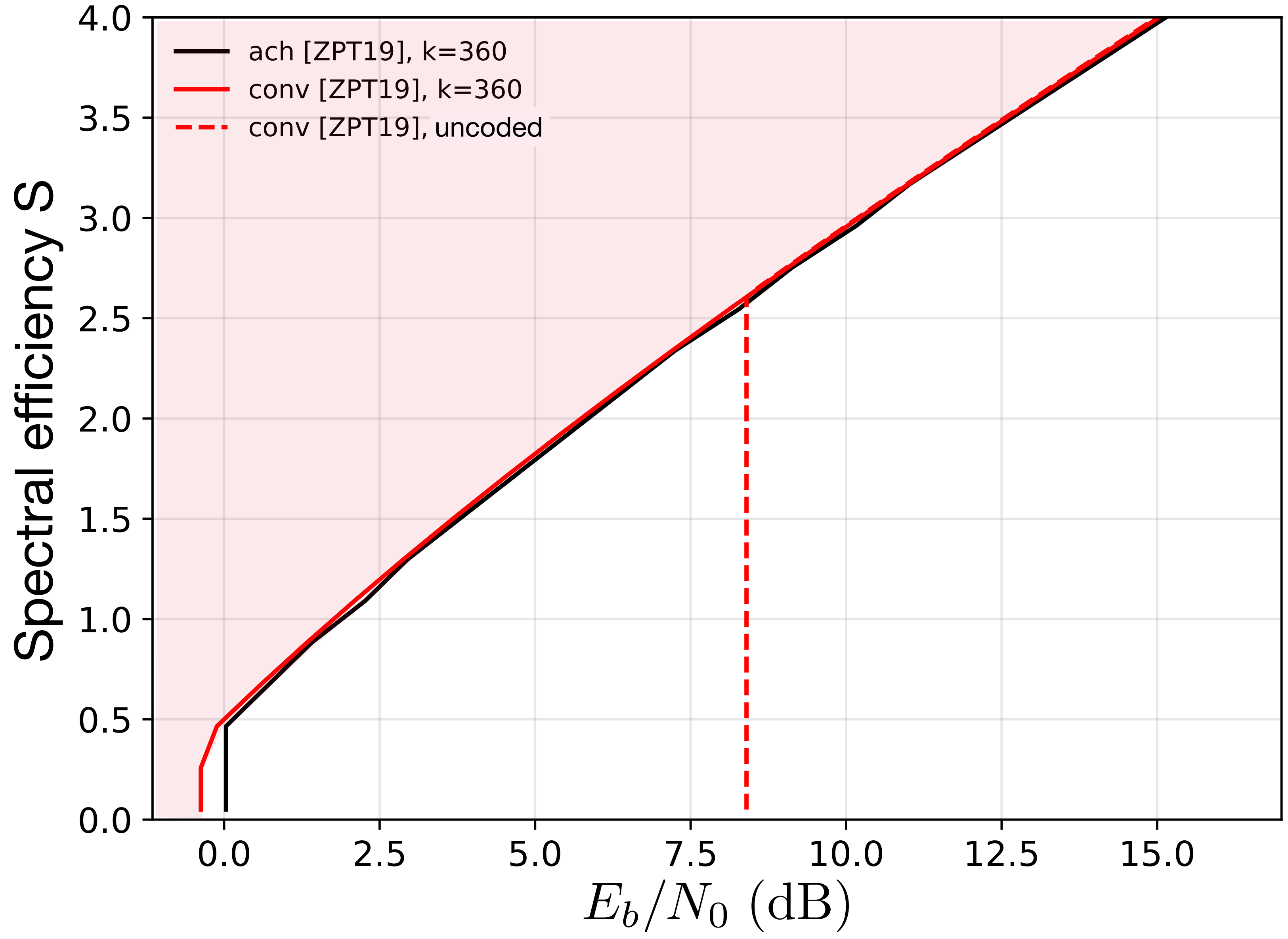

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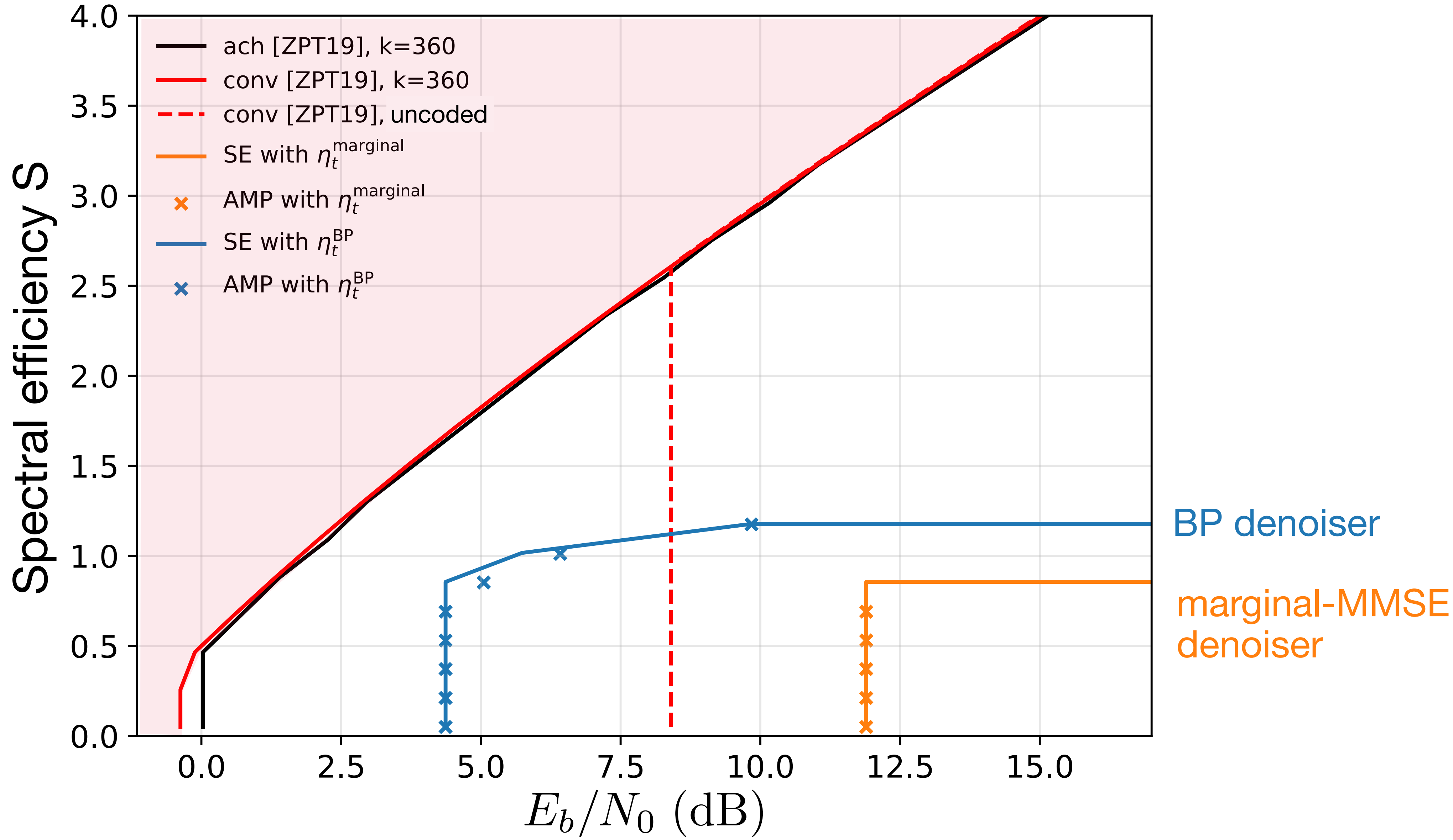

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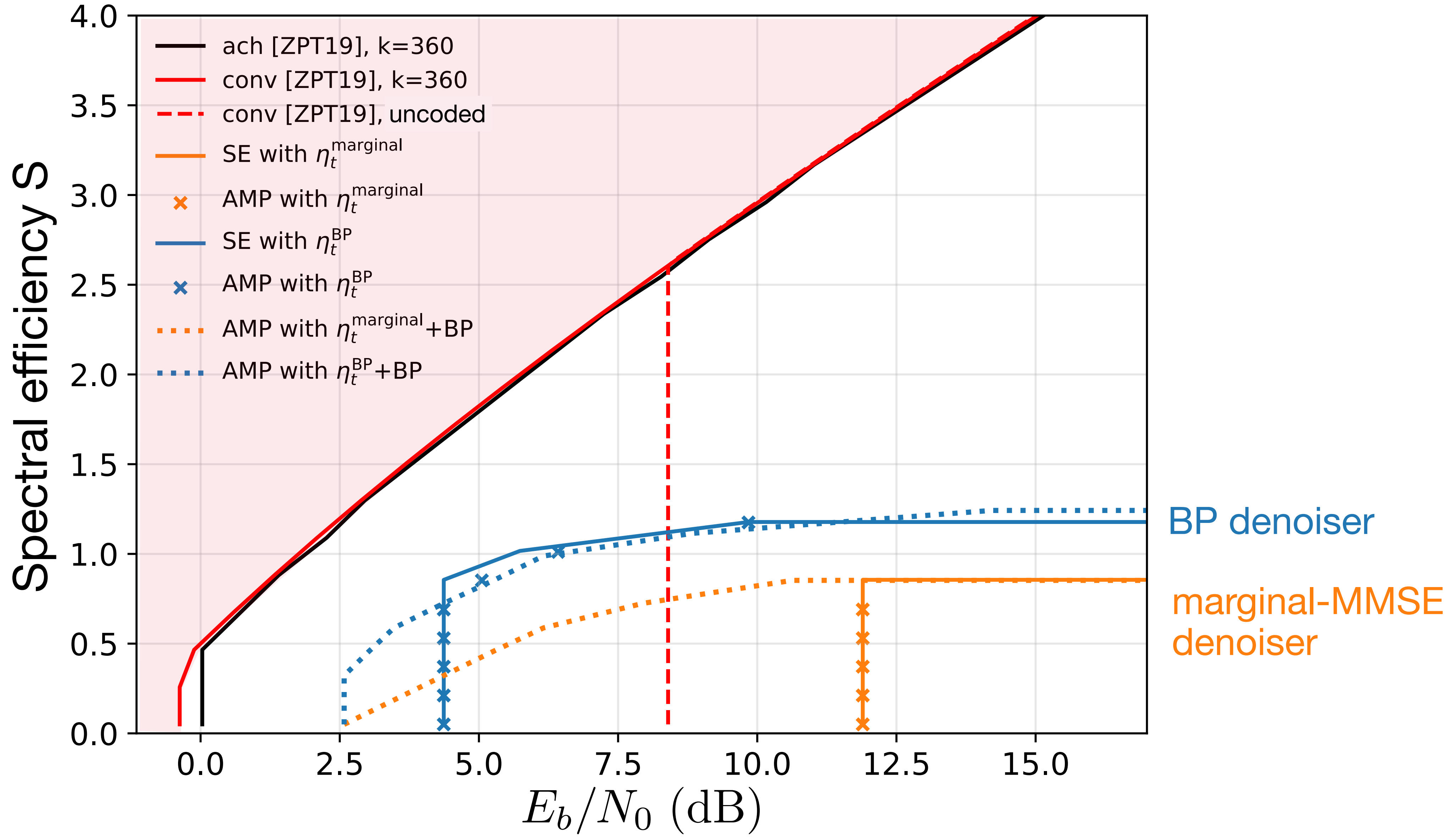

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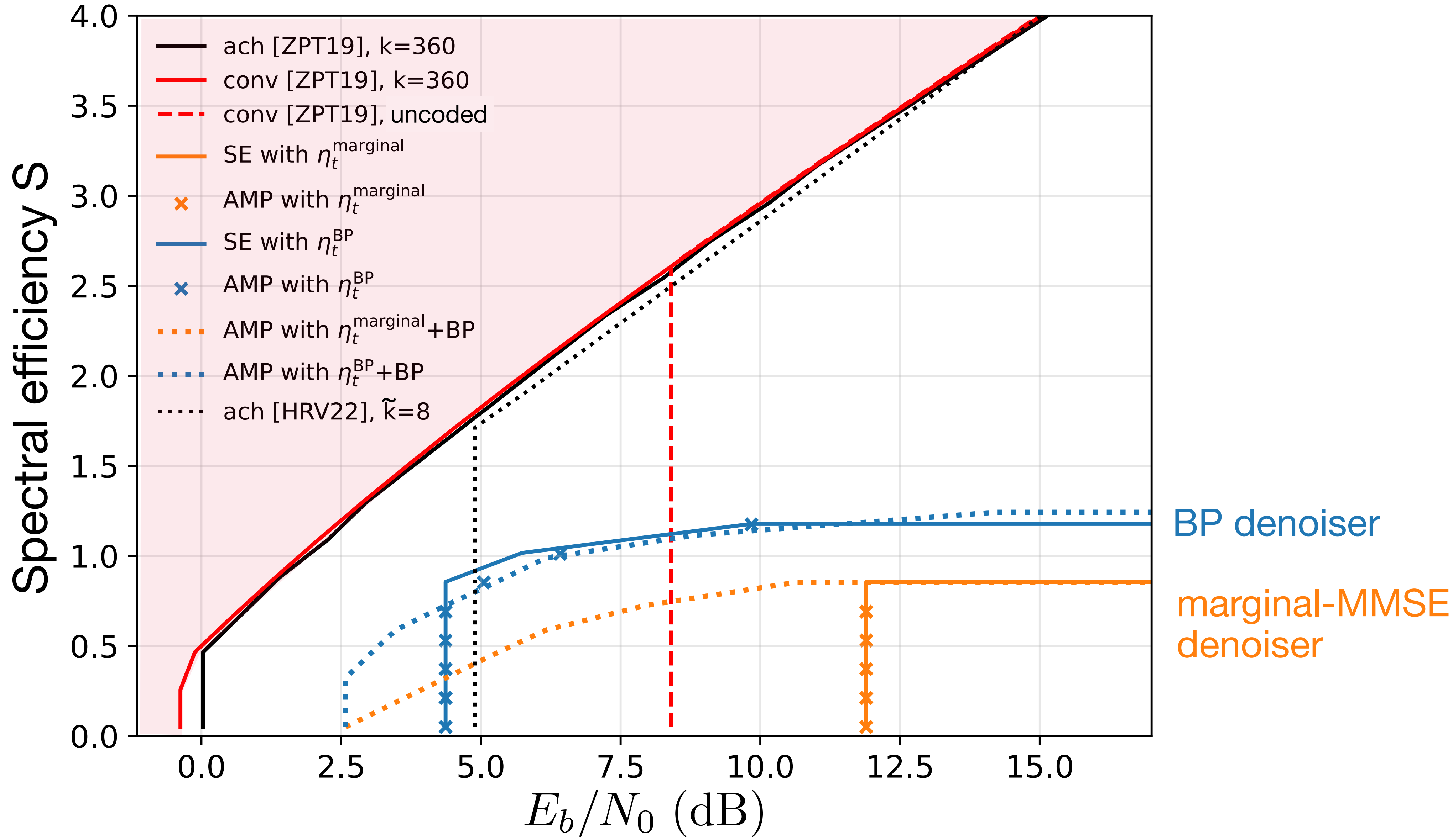

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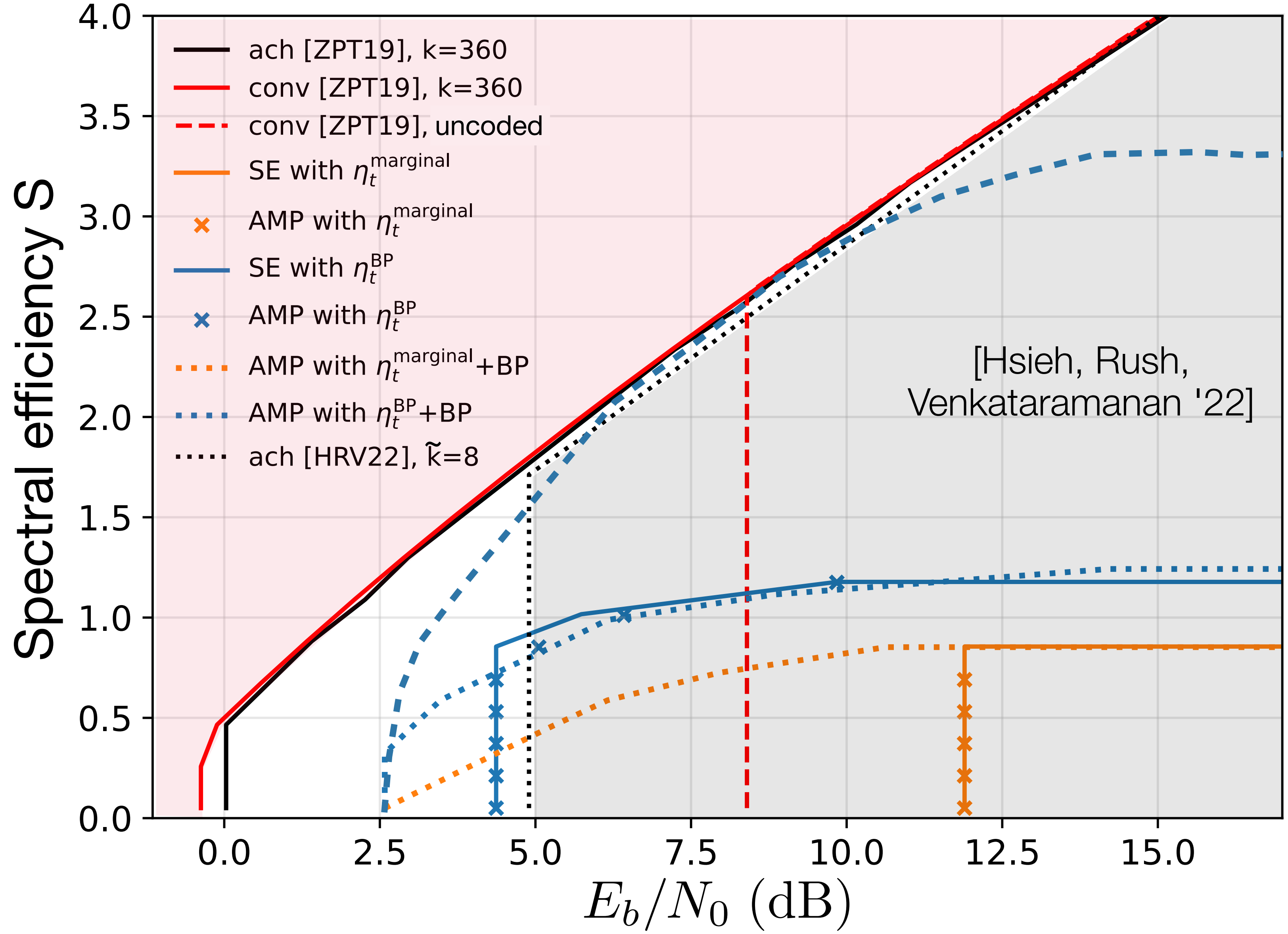
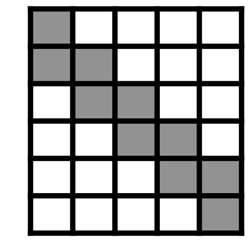


BP denoiser

marginal-MMSE denoiser



SC Gaussian design $\boldsymbol{A}$

Spectral efficiency: total # bits / total # channel uses

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[Liu, Hsieh, Venkataramanan '24]

[Zadik, Polyanskiy, Thrampoulidis '19]

Summary

Many-user Gaussian multiple-access

- State-of-the-art error rates for larger payloads k via

  random binary-CDMA with outer code + AMP decoding with BP denoiser

- Memory and computational costs linear in payload k

- Exact asymptotic error guarantees

- Key future direction: extension to unsourced multiple access

X. Liu, K. Hsieh, and R. Venkataramanan, *Coded many-user multiple access via Approximate Message Passing*, https://arxiv.org/abs/2402.05625, 2024

Correspondence to: {xl394, rv285}@cam.ac.uk

Jamison R. Ebert, Jean-Francois Chamberland, Krishna R. Narayanan, *Multi-User SR-LDPC Codes via Coded Demixing with Applications to Cell-Free Systems*, https://arxiv.org/abs/2402.06881, 2024