Communication over many-user channels via Approximate Message Passing

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Joint work with Ramji Venkataramanan,
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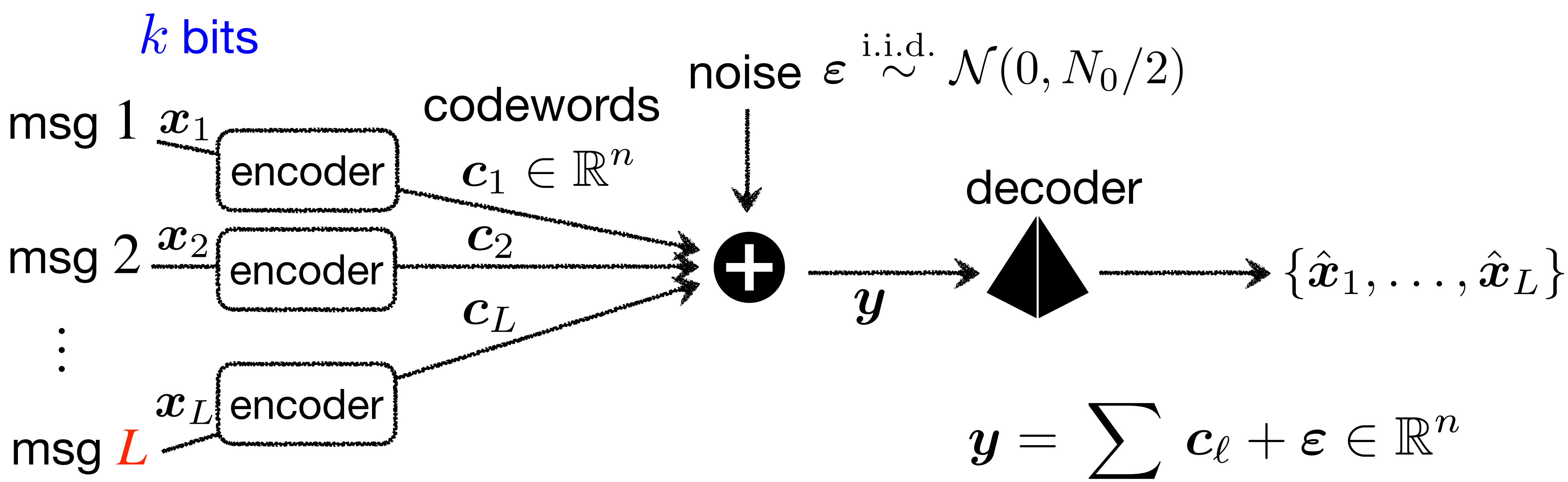
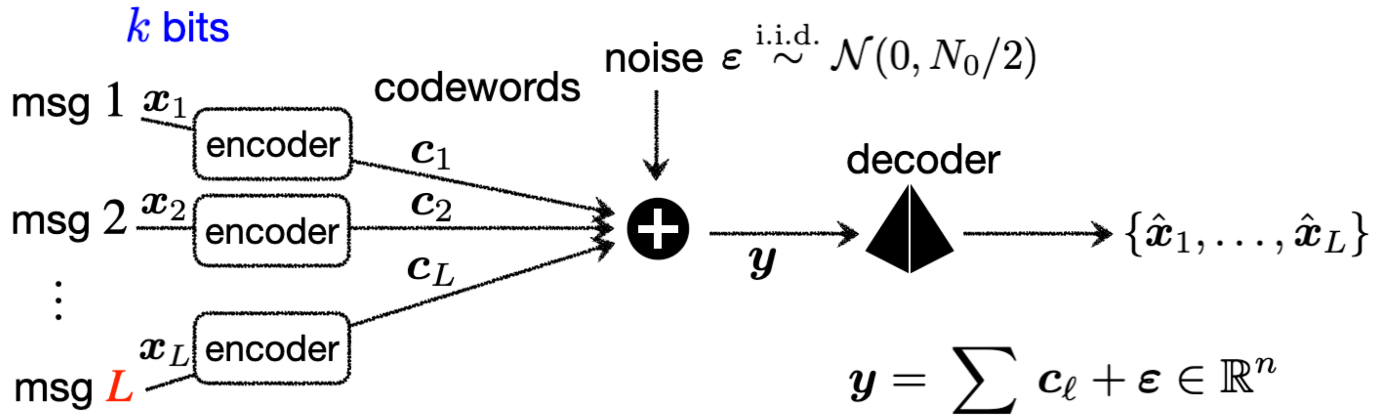
Gaussian multiple-access channel (GMAC)



$$\boldsymbol{y} = \sum_{\ell \in [L]} \boldsymbol{c}_\ell + \boldsymbol{\varepsilon} \in \mathbb{R}^n$$

users have distinct
encoders (i.e. codebooks)

Many-user setting



$$\boldsymbol{y} = \sum_{\ell \in [L]} \boldsymbol{c}_\ell + \boldsymbol{\varepsilon} \in \mathbb{R}^n$$

- user density $\mu := L/n$
- fixed user payload k bits
- energy-per-bit constraint: $\|\boldsymbol{c}_\ell\|_2^2 \leq E := E_b k$
- Per-user probability of error, $\mathrm{PUPE} = \dfrac{1}{L} \sum_{\ell=1}^{L} \mathbb{P}(\hat{\boldsymbol{x}}_\ell \neq \boldsymbol{x}_\ell)$

Linear scaling regime: $L, n \to \infty$ with $\mu := L/n$ fixed

Given $\mu = L/n$, what is minimum E_b/N_0 required to achieve a target PUPE?

[Chen, Chen, Guo, '17], [Ravi, Koch '19], [Polyanskiy '17], [Zadik, Polyanskiy, Thrampoulidis '19], [Polyanskiy, Kowshik '20]

Many-user multiple-access

Previous work [Polyanskiy '17], [Zadik, Polyanskiy, Thrampoulidis '19], [Polyanskiy, Kowshik '20]

What can be achieved without **memory** or **computational** constraints?

random Gaussian codebooks + maximum-likelihood decoding

This talk [Hsieh, Rush, Venkataramanan '22] [Liu, Hsieh, Venkataramanan '24]

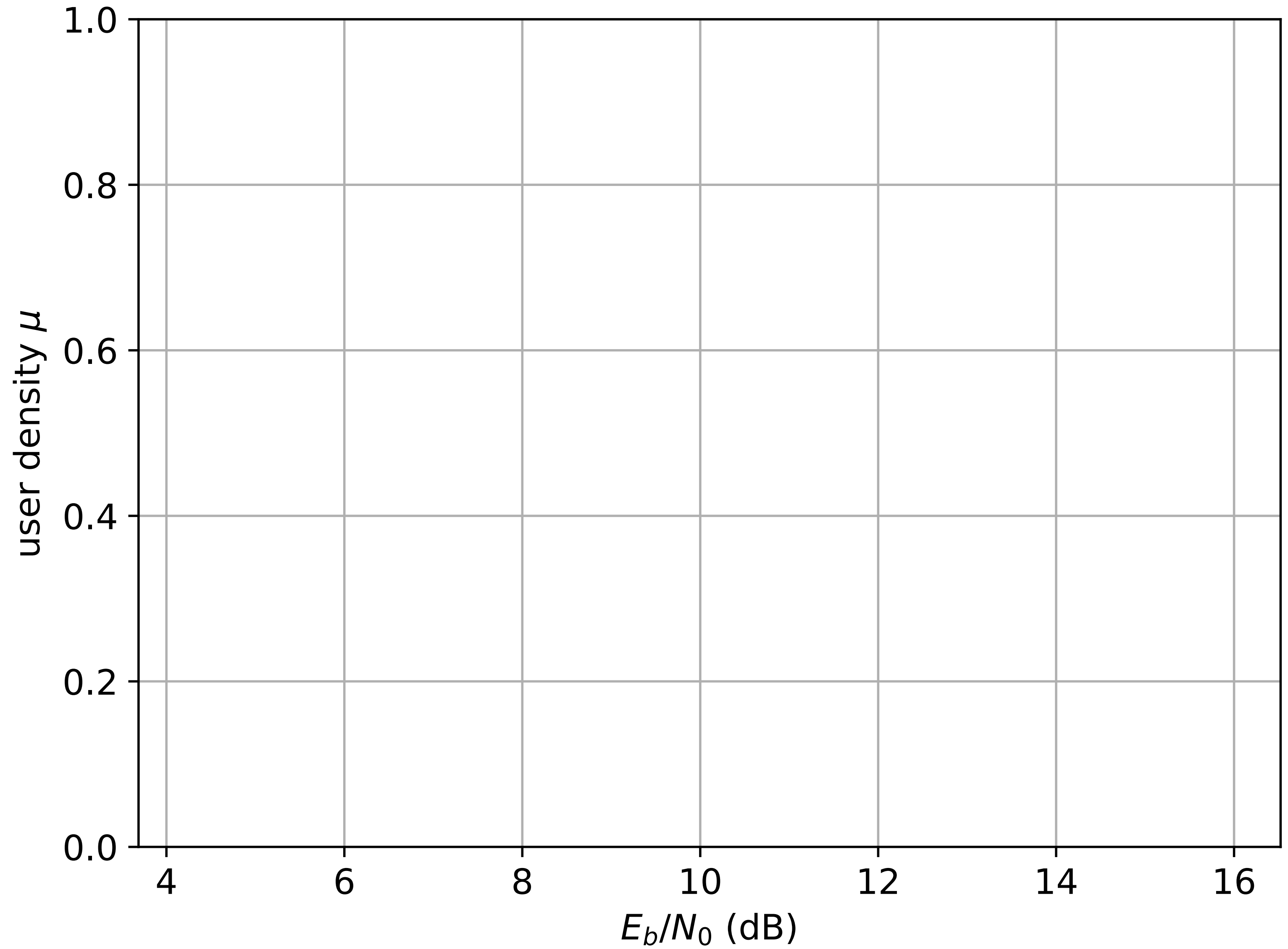
What can be achieved with efficient coding schemes?

random linear coding + Approximate Message Passing (AMP) decoding

Many-user multiple-access with random user activity (last 5mins of talk)

[Liu, Cobo, Venkataramanan '24]

User payload $k = 4$ bits, target PUPE $\leq 10^{-3}$



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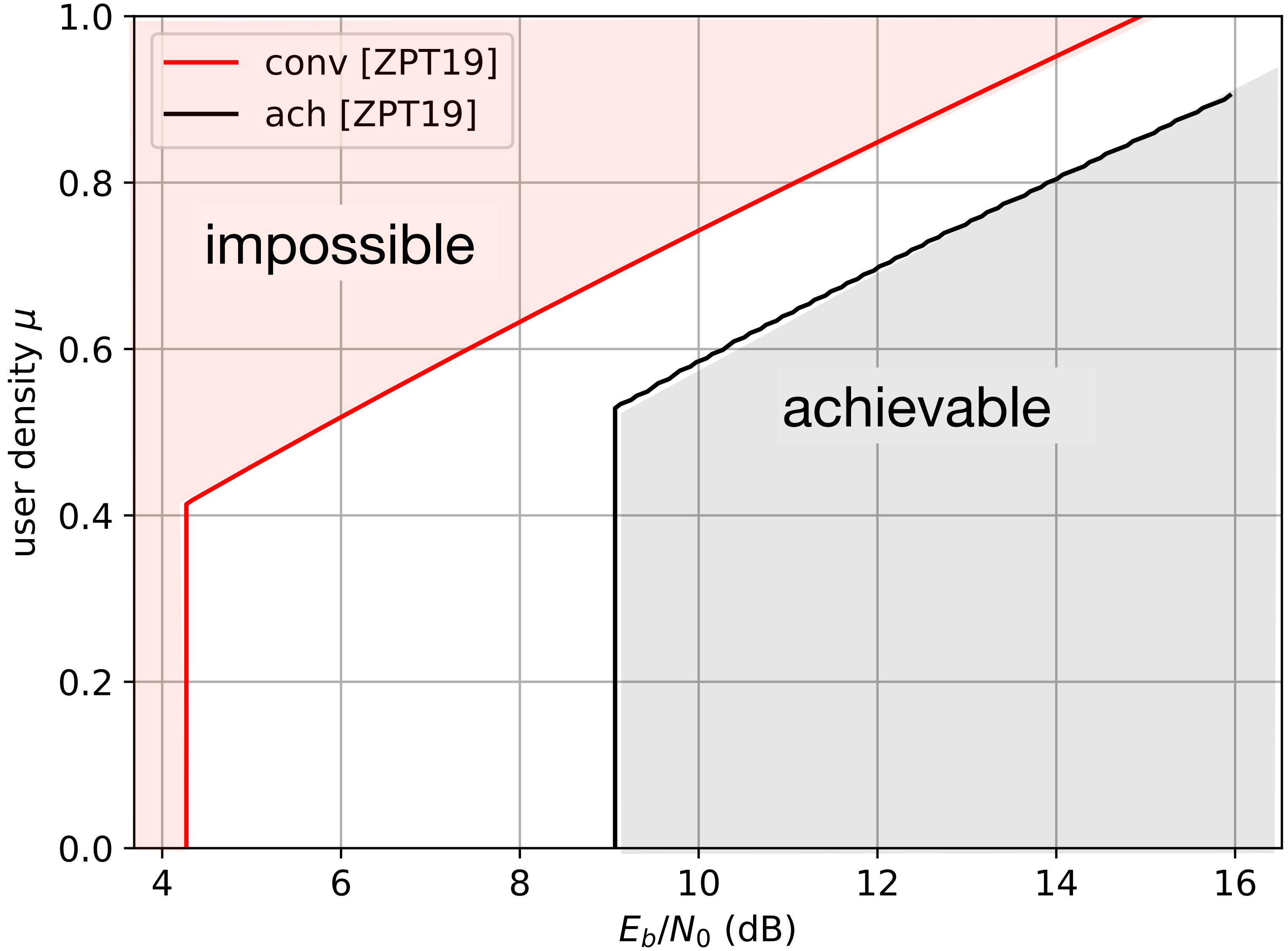

random Gaussian codebooks + ML decoding

[Zadik, Polyanskiy, Thrampoulidis '19]

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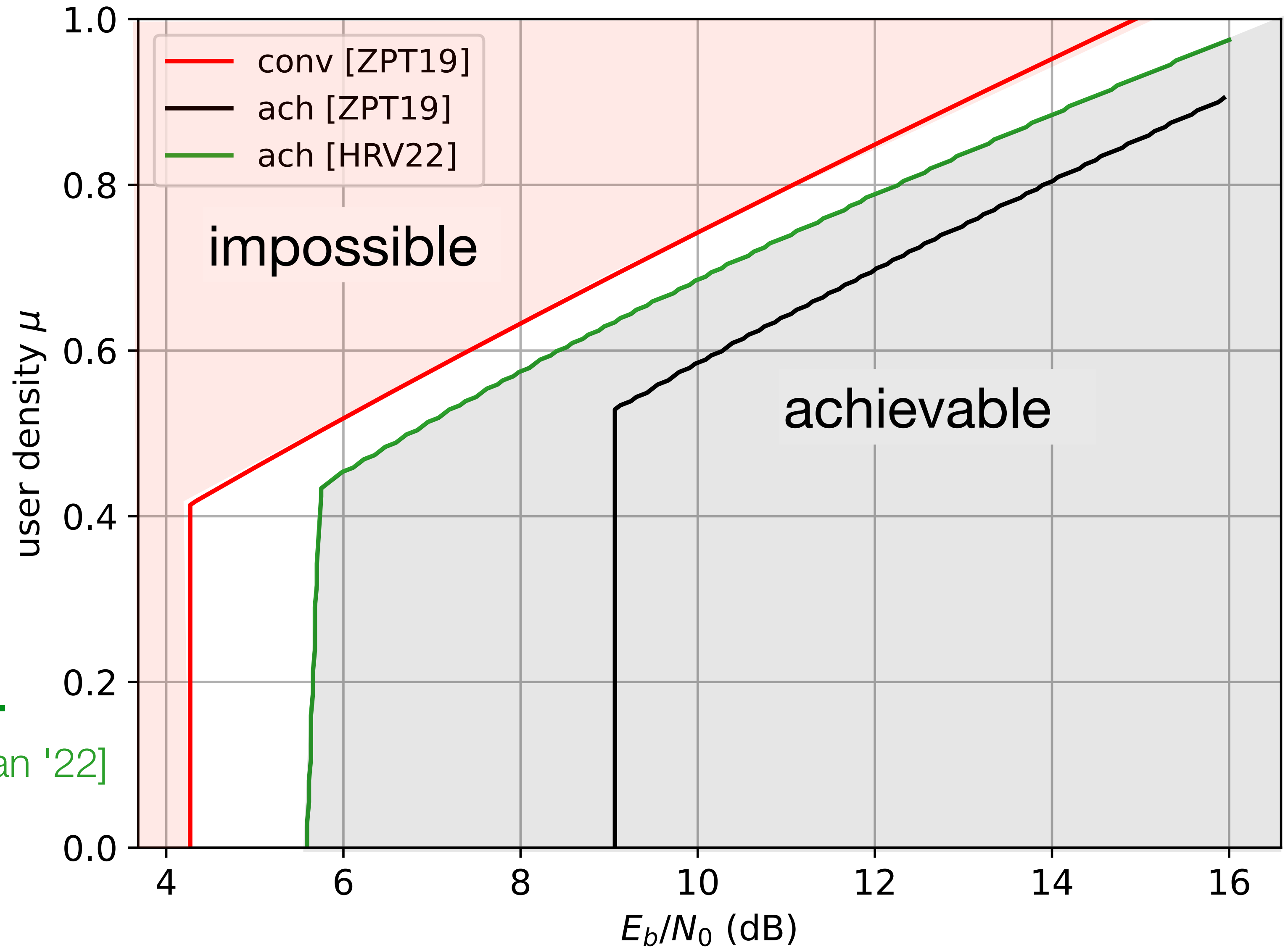


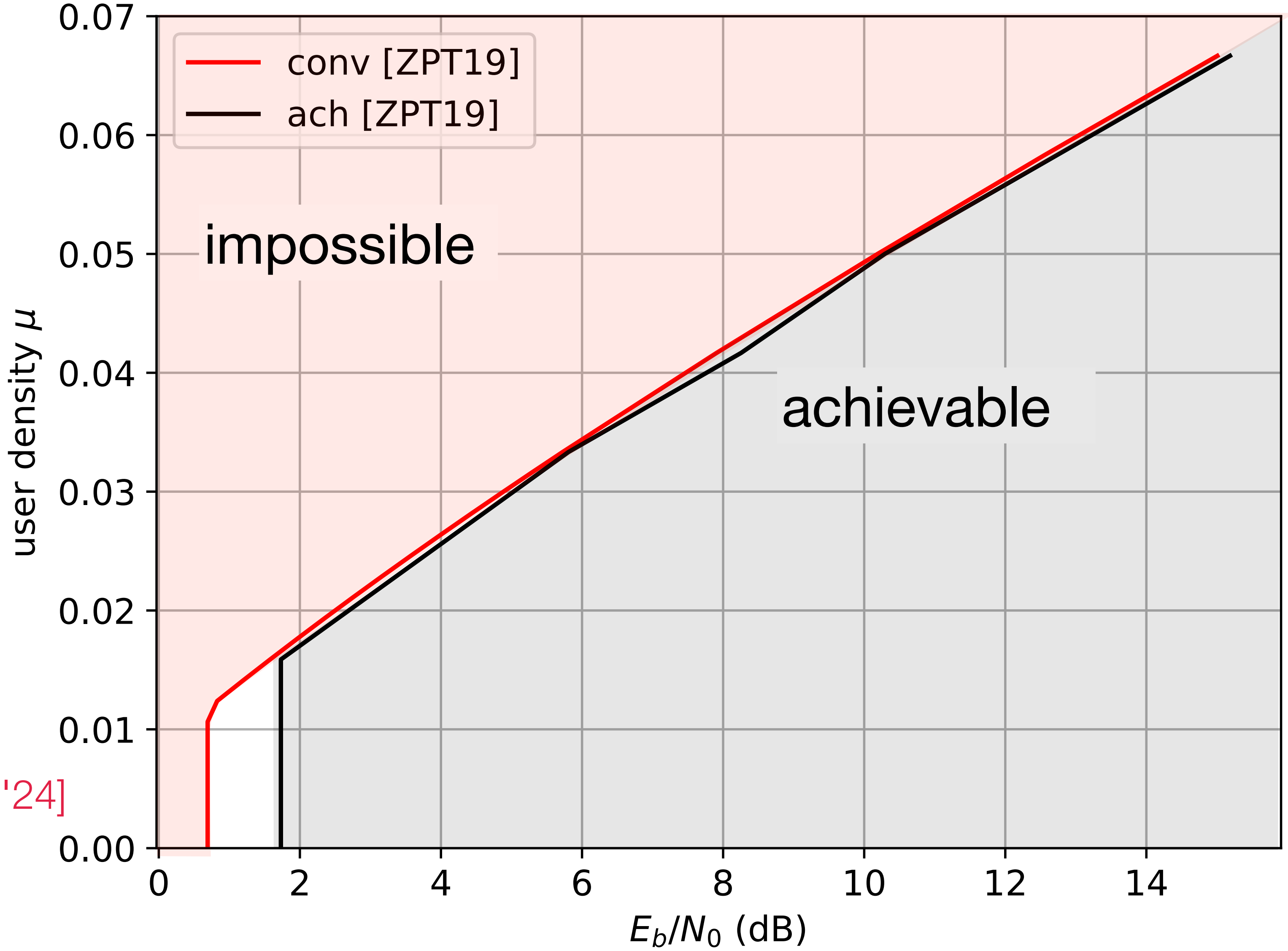
random Gaussian codebooks + ML decoding

Near-optimal practical schemes for small payloads.

[Hsieh, Rush, Venkataramanan '22]

[Zadik, Polyanskiy, Thrampoulidis '19]

User payload $k = 60$ bits, target PUPE $\leq 10^{-3}$



random Gaussian codebooks + ML decoding

Practical schemes for larger payloads?

[Liu, Hsieh, Venkataramanan '24]

[Zadik, Polyanskiy, Thrampoulidis '19]

Practical schemes for small payloads.

[Hsieh, Rush, Venkataramanan '22]

Limitations.

Practical schemes for larger payloads?

[Liu, Hsieh, Venkataramanan '24]

Random linear coding [Hsieh, Rush, Venkataramanan '22]

$$\begin{array}{rcccc} & \text{user } 1 & \text{user } 2 & & \text{user } L \end{array}$$

$$\boldsymbol{A} : \left[\begin{array}{c:c:c:c} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots & \boldsymbol{A}_L \end{array}\right] \updownarrow n$$

$$\boldsymbol{x} : \underbrace{\left[\begin{array}{c:c:c:c} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_L \end{array}\right]^\top}_{B \text{ columns}}$$

- Random matrices: $\boldsymbol{A}_\ell \in \mathbb{R}^{n \times B}$
- User ℓ's k bits payload encoded in $\boldsymbol{x}_\ell \in \mathbb{R}^B \sim p_{\bar{\boldsymbol{x}}}$

  e.g. $B = 2^k$, each $\boldsymbol{x}_\ell$ has a single non-zero entry $= \sqrt{E}$

Each user $\ell \in [L]$ creates codeword $\boldsymbol{c}_\ell = \boldsymbol{A}_\ell \boldsymbol{x}_\ell \in \mathbb{R}^n$

Channel output: $\boldsymbol{y} = \sum_{\ell \in [L]} \boldsymbol{A}_\ell \boldsymbol{x}_\ell + \boldsymbol{\varepsilon} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}$

Random linear coding

[Hsieh, Rush, Venkataraman '22]

$$\begin{array}{c} \\ \boldsymbol{A} : \end{array} \begin{array}{cccc} \text{user } 1 & \text{user } 2 & & \text{user } L \\ \left[\begin{array}{c:c:c:c} \quad \boldsymbol{A}_1 \quad & \quad \boldsymbol{A}_2 \quad & \quad \cdots \quad & \quad \boldsymbol{A}_L \quad \end{array}\right] \end{array} \updownarrow n$$

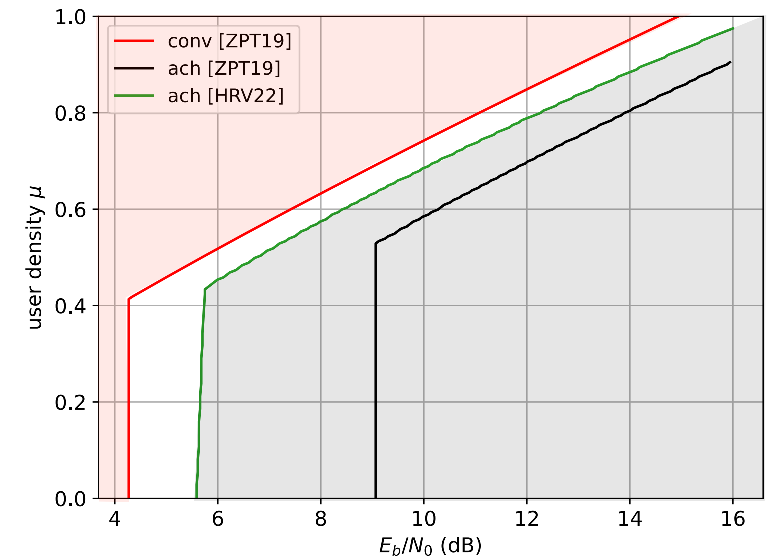
$$\boldsymbol{x} : \left[\begin{array}{c:c:c:c} \quad \boldsymbol{x}_1 \quad & \quad \boldsymbol{x}_2 \quad & \quad \cdots \quad & \quad \boldsymbol{x}_L \quad \end{array}\right]^\top$$

B columns

$$\boldsymbol{y} = \sum_{\ell \in [L]} \boldsymbol{A}_\ell \boldsymbol{x}_\ell + \boldsymbol{\varepsilon} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}$$

Approximate Message Passing (AMP) decoding

Given $\boldsymbol{y}$ and $\boldsymbol{A}$, recover $\boldsymbol{x}$



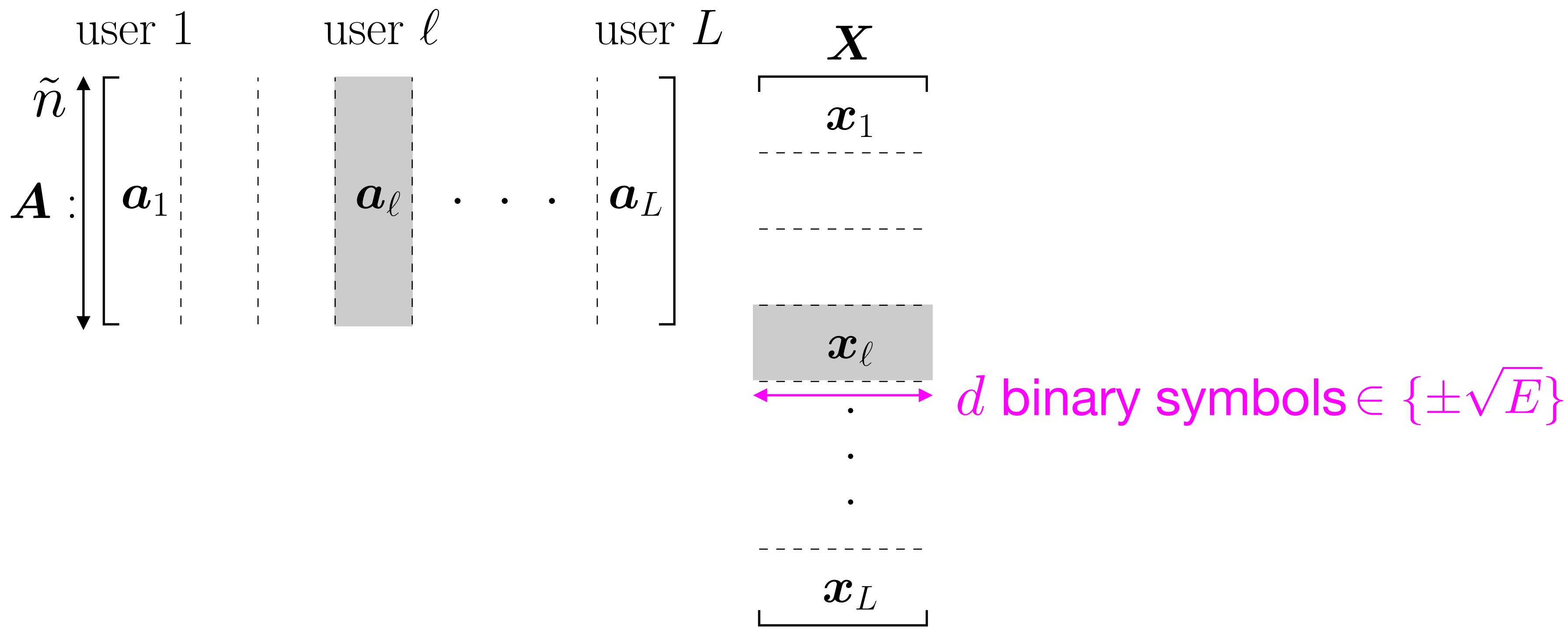
(more on this later)

Take-away: Band-diagonal random design matrix $\boldsymbol{A}$ + AMP decoding gives asymptotic near-optimal error performance for small k

Limitations: memory and computational costs both $\propto 2^k$

Random binary-CDMA + outer code

channel uses
$n = \tilde{n}d$



For each user $\ell \in [L]$:

- Random signature sequence: $\boldsymbol{a}_\ell \in \mathbb{R}^{\tilde{n}}$
- Outer (k,d) linear code: k bits payload encoded in $\boldsymbol{x}_\ell \in \{\pm\sqrt{E}\}^d \sim p_{\bar{\boldsymbol{x}}}$
  (e.g. LDPC code)

Channel output: $\boldsymbol{Y} = \sum_{\ell\in[L]} \boldsymbol{a}_\ell \boldsymbol{x}_\ell + \boldsymbol{\mathcal{E}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{\mathcal{E}} \in \mathbb{R}^{\tilde{n}\times d}$

Benefits: Memory and computational costs linear in k

Challenge: efficiently decode outer code?

[Liu, Hsieh, Venkataramanan '24]

AMP decoder for i.i.d. Gaussian $\boldsymbol{A}$

Given $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{\mathcal{E}} \in \mathbb{R}^{\tilde{n}\times d}$ and $\boldsymbol{A}$, recover $\boldsymbol{X}$

Start with initialiser $\boldsymbol{X}^0 = \mathbf{0}$, for $t \geq 1$

debias term

$$\boldsymbol{Z}^t = \boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}^t + \boxed{\frac{1}{\tilde{n}}\boldsymbol{Z}^{t-1}\Big[\sum_{\ell\in[L]} \eta'_{t-1}\left(\boldsymbol{s}_\ell^{t-1}\right)\Big]^\top}$$

effective observation

$$\boldsymbol{X}^{t+1} = \eta_t(\boldsymbol{S}^t), \quad \boldsymbol{S}^t = \boldsymbol{A}^\top \boldsymbol{Z}^t + \boldsymbol{X}^t$$

End with hard decision estimate $\hat{\boldsymbol{X}}^{t+1} = h_t(\boldsymbol{S}^t)$

- $\eta_t : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz and applies **row-wise** to matrix inputs
- debias term ensures empirical dist. of rows of $(\boldsymbol{S}^t - \boldsymbol{X}) \stackrel{d}{=} \boldsymbol{g}^t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}^t)$
- η_t estimates $\boldsymbol{X}$ from observation in Gaussian noise

can be deterministically computed

AMP decoder

$$\boldsymbol{Z}^t = \boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}^t + \frac{1}{\tilde{n}}\boldsymbol{Z}^{t-1}\Big[\sum_{\ell\in[L]} \eta'_{t-1}\left(\boldsymbol{s}_\ell^{t-1}\right)\Big]^\top$$

$$\boldsymbol{X}^{t+1} = \eta_t\left(\boldsymbol{S}^t\right), \quad \boldsymbol{S}^t = \boldsymbol{A}^\top \boldsymbol{Z}^t + \boldsymbol{X}^t \qquad (\boldsymbol{s}_\ell^t - \boldsymbol{x}_\ell) \stackrel{d}{=} \boldsymbol{g}^t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}^t)$$

Hard decision estimate: $\hat{\boldsymbol{X}}^{t+1} = h_t(\boldsymbol{S}^t)$

Theorem [Liu, Hsieh, Venkataramanan '24]

For $\boldsymbol{A} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1/\tilde{n})$, let $\eta_1, \dots, \eta_t$ be Lipschitz, then

Asymp. user error rate (UER)

$$\frac{1}{L}\sum_{\ell=1}^{L} \mathbb{1}\{\hat{\boldsymbol{x}}_\ell^{t+1} \neq \boldsymbol{x}_\ell\}$$

Asymp. bit error rate (BER)

$$\frac{1}{Ld}\sum_{\ell=1}^{L}\sum_{i=1}^{d} \mathbb{1}\left\{\hat{x}_{\ell,i}^{t+1} \neq x_{\ell,i}\right\}$$

AMP decoder

$$\boldsymbol{Z}^t = \boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}^t + \frac{1}{\tilde{n}}\boldsymbol{Z}^{t-1}\Big[\sum_{\ell\in[L]} \eta'_{t-1}\left(\boldsymbol{s}_\ell^{t-1}\right)\Big]^\top$$

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<u>Asymp. user error rate (UER)</u>

$$\lim_{L\to\infty} \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{1}\{\hat{\boldsymbol{x}}_\ell^{t+1} \neq \boldsymbol{x}_\ell\} = \boxed{\mathbb{P}\left(h_t\left(\bar{\boldsymbol{x}} + \boldsymbol{g}^t\right) \neq \bar{\boldsymbol{x}}\right)}$$

deterministic

<u>Asymp. bit error rate (BER)</u>

$$\lim_{L\to\infty} \frac{1}{Ld}\sum_{\ell=1}^{L}\sum_{i=1}^{d} \mathbb{1}\left\{\hat{x}_{\ell,i}^{t+1} \neq x_{\ell,i}\right\} = \boxed{\frac{1}{d}\sum_{i=1}^{d} \mathbb{P}\left(\left[h_t\left(\bar{x} + g^t\right)\right]_i \neq \bar{x}_i\right)}$$

- $\bar{\boldsymbol{x}} \in \{\pm\sqrt{E}\}^d$ uniformly distributed among 2^k codewords
- $\boldsymbol{g}^t \in \mathbb{R}^d \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}^t)$ independent of $\bar{\boldsymbol{x}}$

AMP decoder

$$\boldsymbol{Z}^t = \boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}^t + \frac{1}{\tilde{n}}\boldsymbol{Z}^{t-1}\Big[\sum_{\ell\in[L]} \eta'_{t-1}\left(\boldsymbol{s}_\ell^{t-1}\right)\Big]^\top$$

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Hard decision estimate: $\hat{\boldsymbol{X}}^{t+1} = h_t(\boldsymbol{S}^t)$

How to choose denoiser $\eta_t : \mathbb{R}^d \to \mathbb{R}^d$?

- **Bayes-optimal**

computational cost

$$\boldsymbol{x}_\ell^{t+1} = \eta_t(\boldsymbol{s}_\ell^t) = \mathbb{E}\left[\bar{\boldsymbol{x}} \mid \bar{\boldsymbol{x}} + \boldsymbol{g}^t = \boldsymbol{s}_\ell^t\right] \qquad O(2^k d^3)$$

$$= \sum_{\boldsymbol{x}'} \boldsymbol{x}' \cdot \frac{\exp\left(-\frac{1}{2}(\boldsymbol{x}' - 2\boldsymbol{s}_\ell^t)^\top (\boldsymbol{\Sigma}^t)^{-1}\boldsymbol{x}'\right)}{\sum_{\tilde{\boldsymbol{x}}'} \exp\left(-\frac{1}{2}(\tilde{\boldsymbol{x}}' - 2\boldsymbol{s}_\ell^t)^\top (\boldsymbol{\Sigma}^t)^{-1}\tilde{\boldsymbol{x}}'\right)}$$

- **marginal-MMSE**

$$\boldsymbol{x}_\ell^{t+1} = \eta_t(\boldsymbol{s}_\ell^t) = \begin{bmatrix} \mathbb{E}[\bar{x}_1 \mid \bar{x}_1 + g_1^t = s_{\ell,1}^t] \\ \vdots \\ \mathbb{E}[\bar{x}_d \mid \bar{x}_d + g_d^t = s_{\ell,d}^t] \end{bmatrix} \qquad O(d)$$

- **Belief Propagation (BP)** [Amalladine et al. '22], [Ebert et al. '23]

$$\boldsymbol{x}_\ell^{t+1} = \eta_t(\boldsymbol{s}_\ell^t) \approx \begin{bmatrix} \mathbb{E}[\bar{x}_1 \mid \bar{x}_1 + g_1^t = s_{\ell,1}^t \text{ , parities involving } \bar{x}_1 \text{ satisfied}] \\ \vdots \\ \mathbb{E}[\bar{x}_d \mid \bar{x}_d + g_d^t = s_{\ell,d}^t \text{ , parities involving } \bar{x}_d \text{ satisfied}] \end{bmatrix} \qquad O(d)$$

AMP decoder

$$\boldsymbol{Z}^t = \boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}^t + \boxed{\frac{1}{\tilde{n}}\boldsymbol{Z}^{t-1}\Big[\sum_{\ell\in[L]} \eta'_{t-1}\big(\boldsymbol{s}_\ell^{t-1}\big)\Big]^\top}$$ debias term

$$\boldsymbol{X}^{t+1} = \eta_t\big(\boldsymbol{S}^t\big), \quad \boldsymbol{S}^t = \boldsymbol{A}^\top \boldsymbol{Z}^t + \boldsymbol{X}^t \qquad (\boldsymbol{s}_\ell^t - \boldsymbol{x}_\ell) \stackrel{d}{=} \boldsymbol{g}^t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}^t)$$

Hard decision estimate: $\hat{\boldsymbol{X}}^{t+1} = h_t(\boldsymbol{S}^t)$

How to compute Jacobian η'_t ?

- **Belief Propagation (BP)** [Amalladine et al. '22], [Ebert et al. '23]

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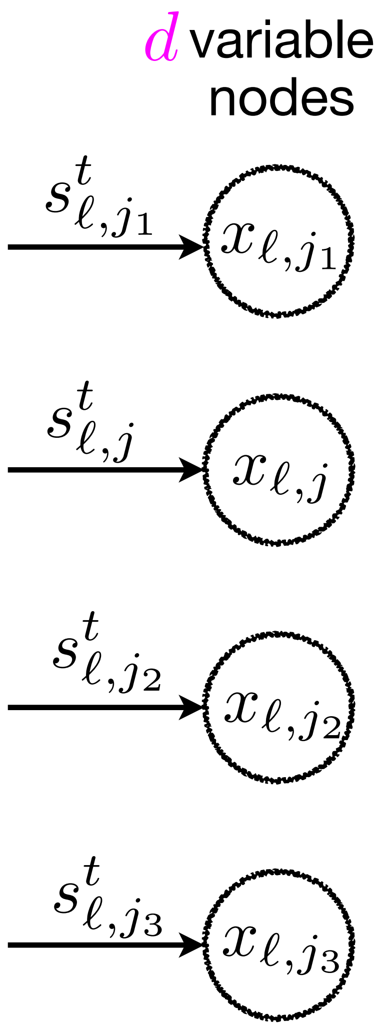
Belief Propagation (BP) denoiser $\boldsymbol{x}_\ell^{t+1} = \eta_t(\boldsymbol{s}_\ell^t)$



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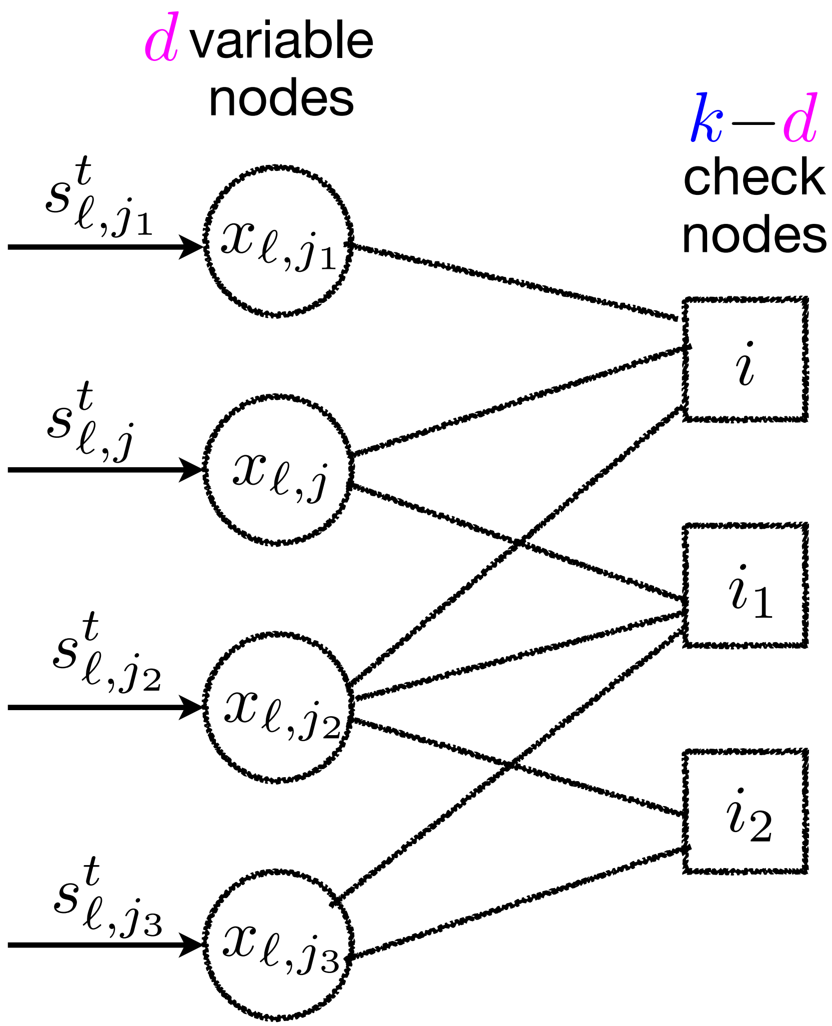

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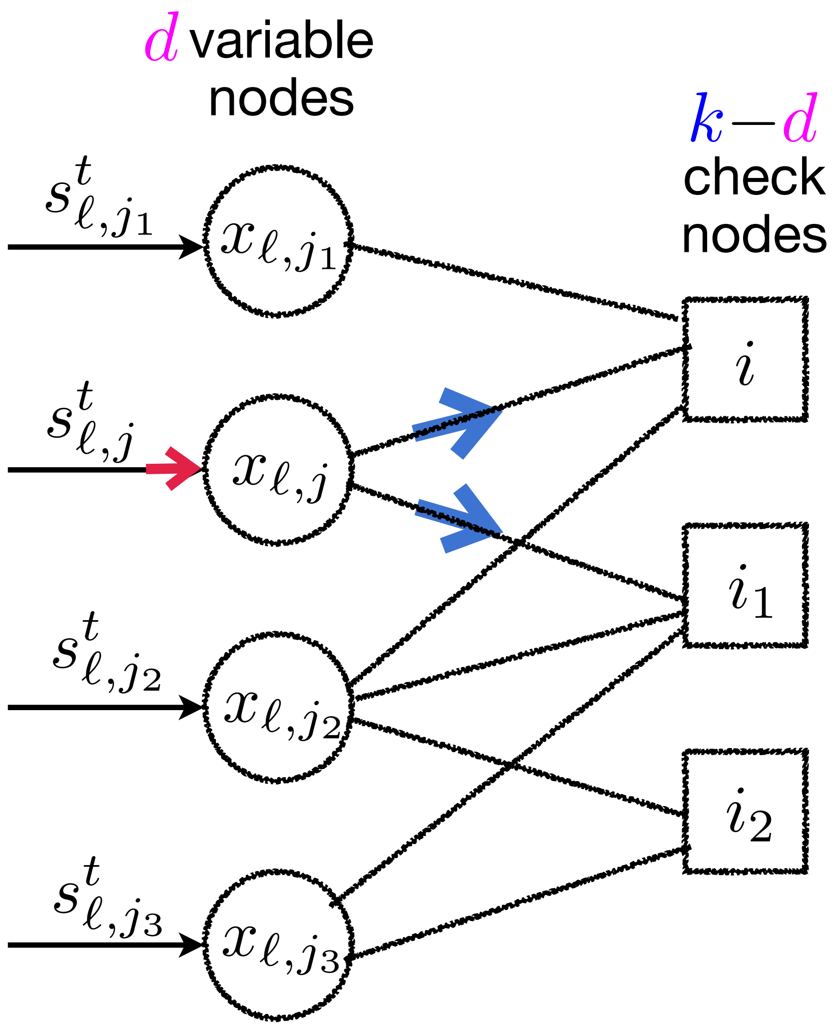

1. Initialise:

$$L^{(0)}_{j\to i} = \ln\left[\frac{p(s^t_{\ell,j}\,|x_{\ell,j} = +\sqrt{E})}{p(s^t_{\ell,j}\,|x_{\ell,j} = -\sqrt{E})}\right] = \frac{2\sqrt{E}s^t_{\ell,j}}{\Sigma^t_{j,j}} =: \underline{L(s^t_{\ell,j})}$$

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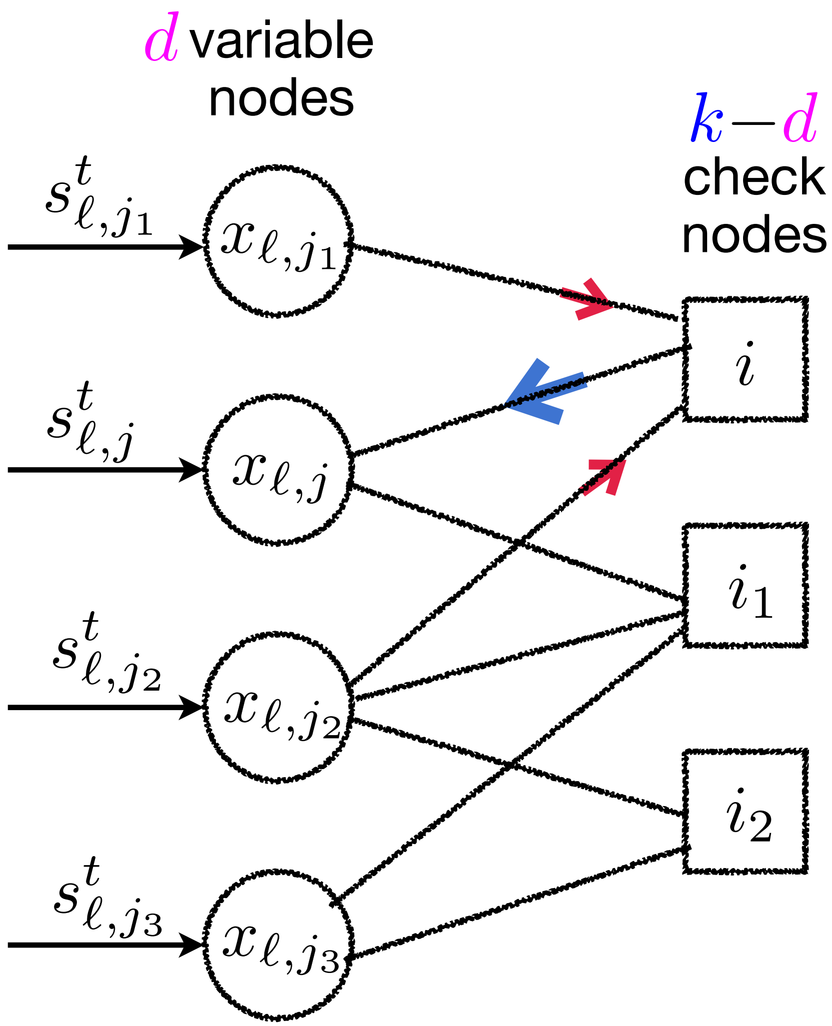

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2. Middle rounds:

$$L^{(r)}_{i\to j} = 2\tanh^{-1}\left[\prod_{j'\in N(i)\setminus j} \tanh\left(\frac{1}{2}L^{(r-1)}_{j'\to i}\right)\right]$$

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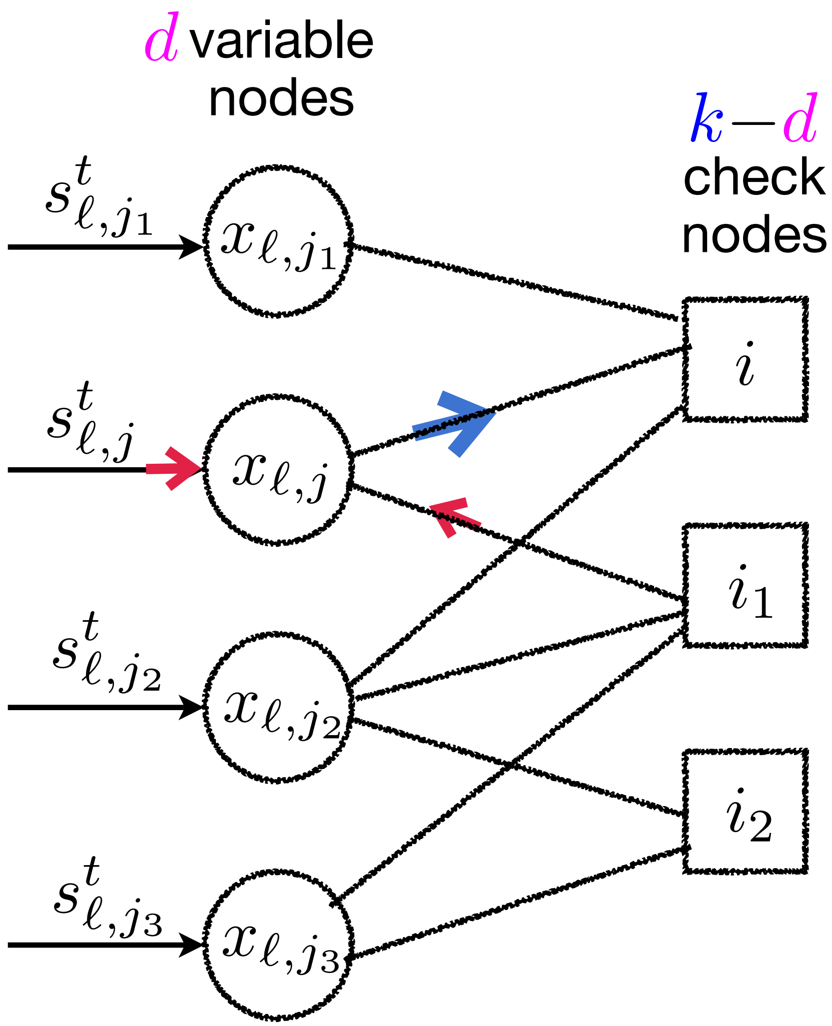

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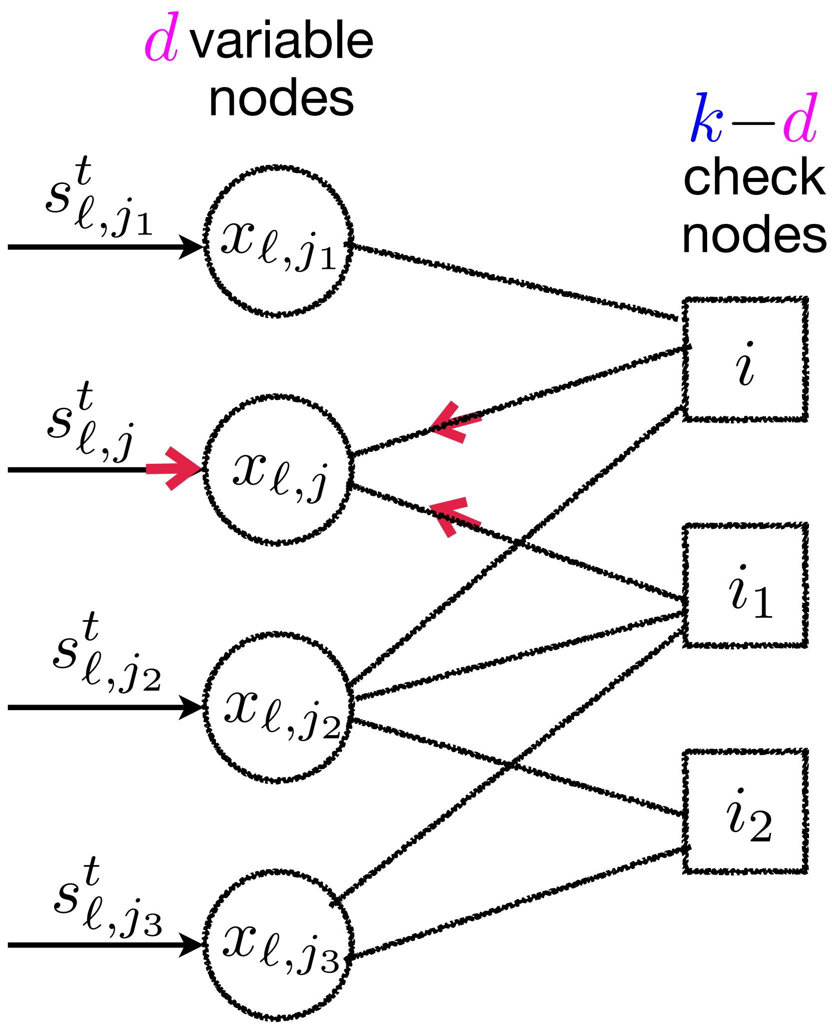

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3. Terminate after $\mathcal{R}$ rounds:

$$L^{(\mathcal{R})}_j = L(s^t_{\ell,j}) + \sum_{i\in N(j)} L^{(\mathcal{R})}_{i\to j}$$

approx. marginal posterior probabilities

4. Update AMP estimate:

$$\left[\eta_t(\boldsymbol{s}^t_\ell)\right]_j = \sqrt{E}\ \tanh\left(L^{(\mathcal{R})}_j/2\right)$$
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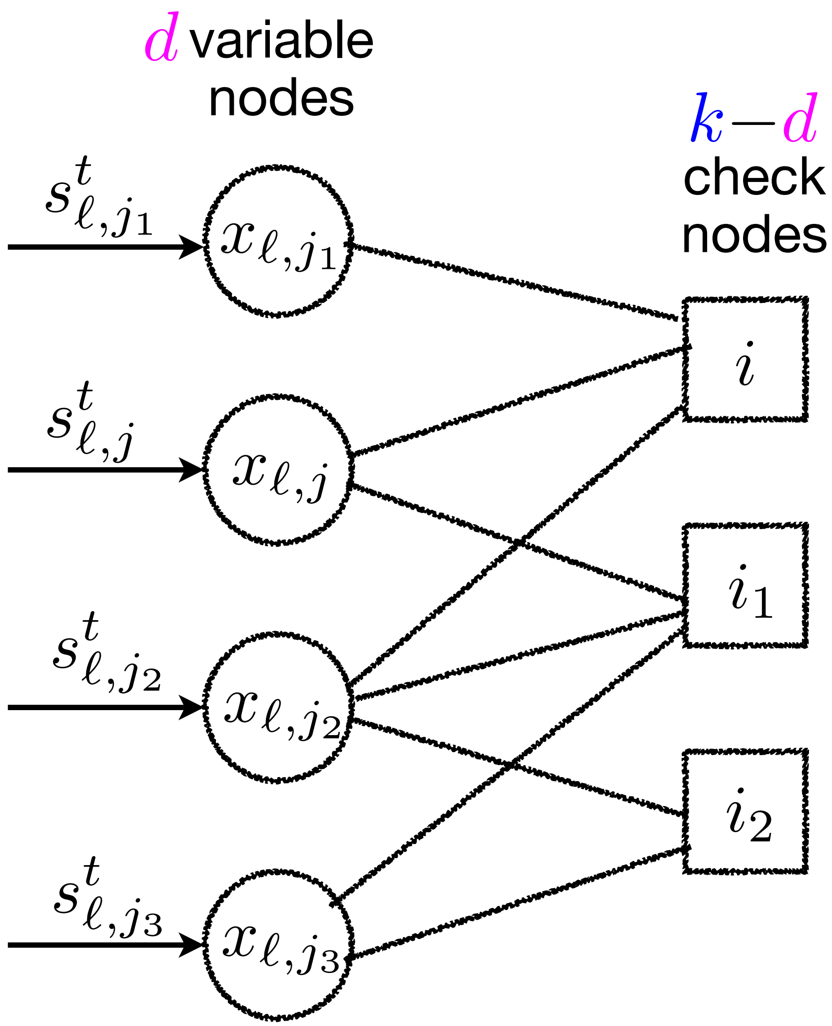

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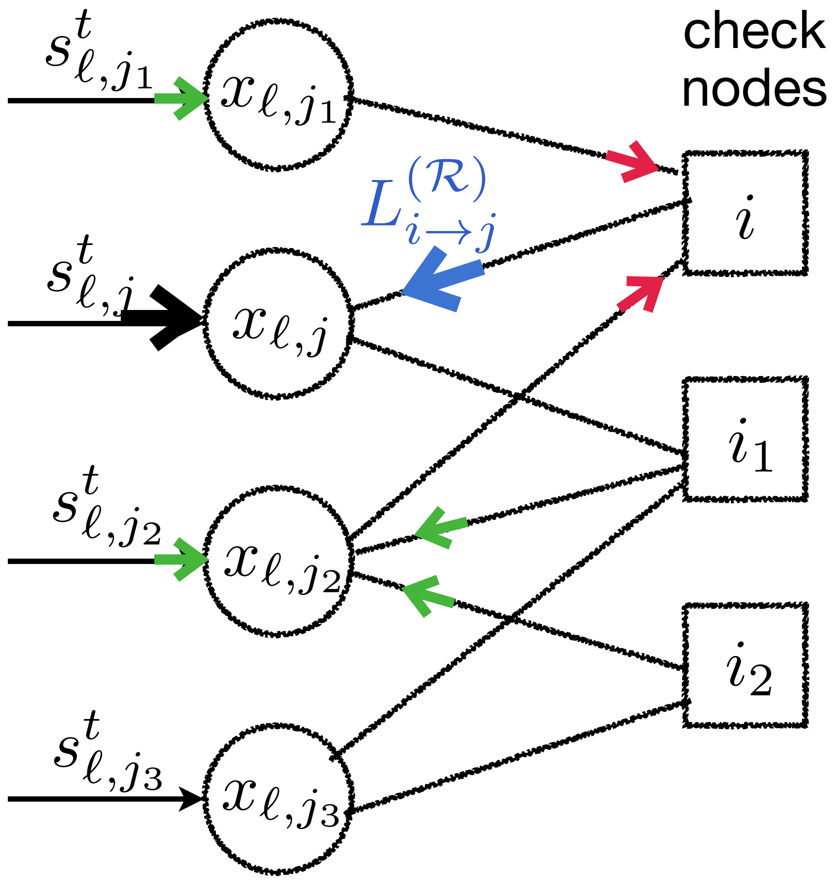
Jacobian has closed form expression when $\mathcal{R}$ < girth of bipartite graph!

[Amalladine et al. '22], [Ebert et al. '23]

Belief Propagation (BP) denoiser $x_\ell^{t+1} = \eta_t(s_\ell^t)$



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[Amalladine et al. '22], [Ebert et al. '23]

Theorem for i.i.d. Gaussian $\boldsymbol{A}$ [Liu, Hsieh, Venkataramanan '24]

For $\boldsymbol{A} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1/\tilde{n})$, let $\eta_1, \dots, \eta_t$ be Lipschitz, then

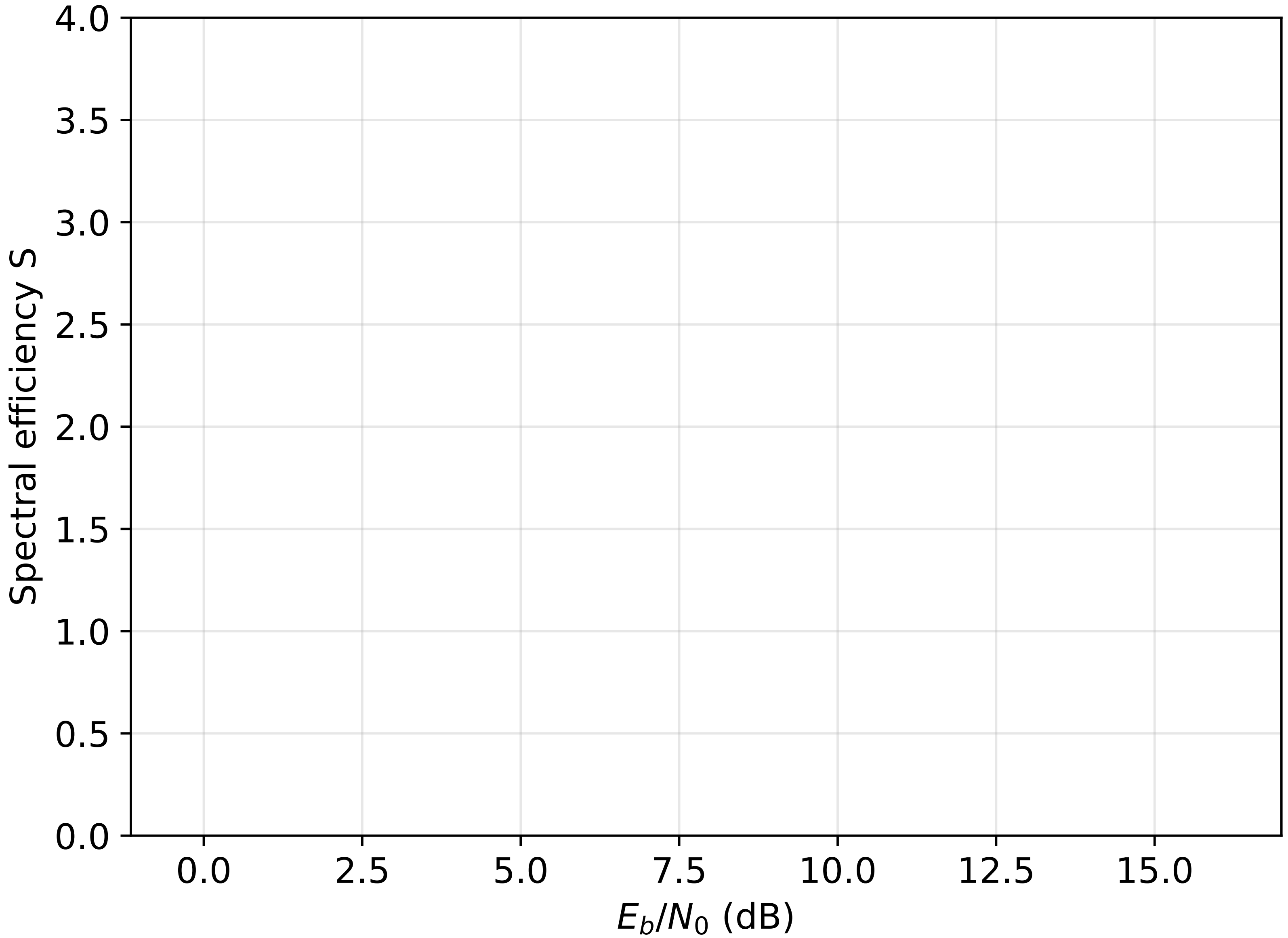
$$\lim_{L\to\infty} \texttt{UER} := \lim_{L\to\infty} \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{1}\{\hat{\boldsymbol{x}}_\ell^{t+1} \neq \boldsymbol{x}_\ell\} = \boxed{\mathbb{P}\left(h_t\left(\bar{\boldsymbol{x}} + \boldsymbol{g}^t\right) \neq \bar{\boldsymbol{x}}\right)}$$

deterministic

$$\lim_{L\to\infty} \texttt{BER} := \lim_{L\to\infty} \frac{1}{Ld}\sum_{\ell=1}^{L}\sum_{i=1}^{d} \mathbb{1}\left\{\hat{x}_{\ell,i}^{t+1} \neq x_{\ell,i}\right\} = \boxed{\frac{1}{d}\sum_{i=1}^{d} \mathbb{P}\left(\left[h_t\left(\bar{x} + g^t\right)\right]_i \neq \bar{x}_i\right)}.$$

Applies to AMP with BP denoiser!

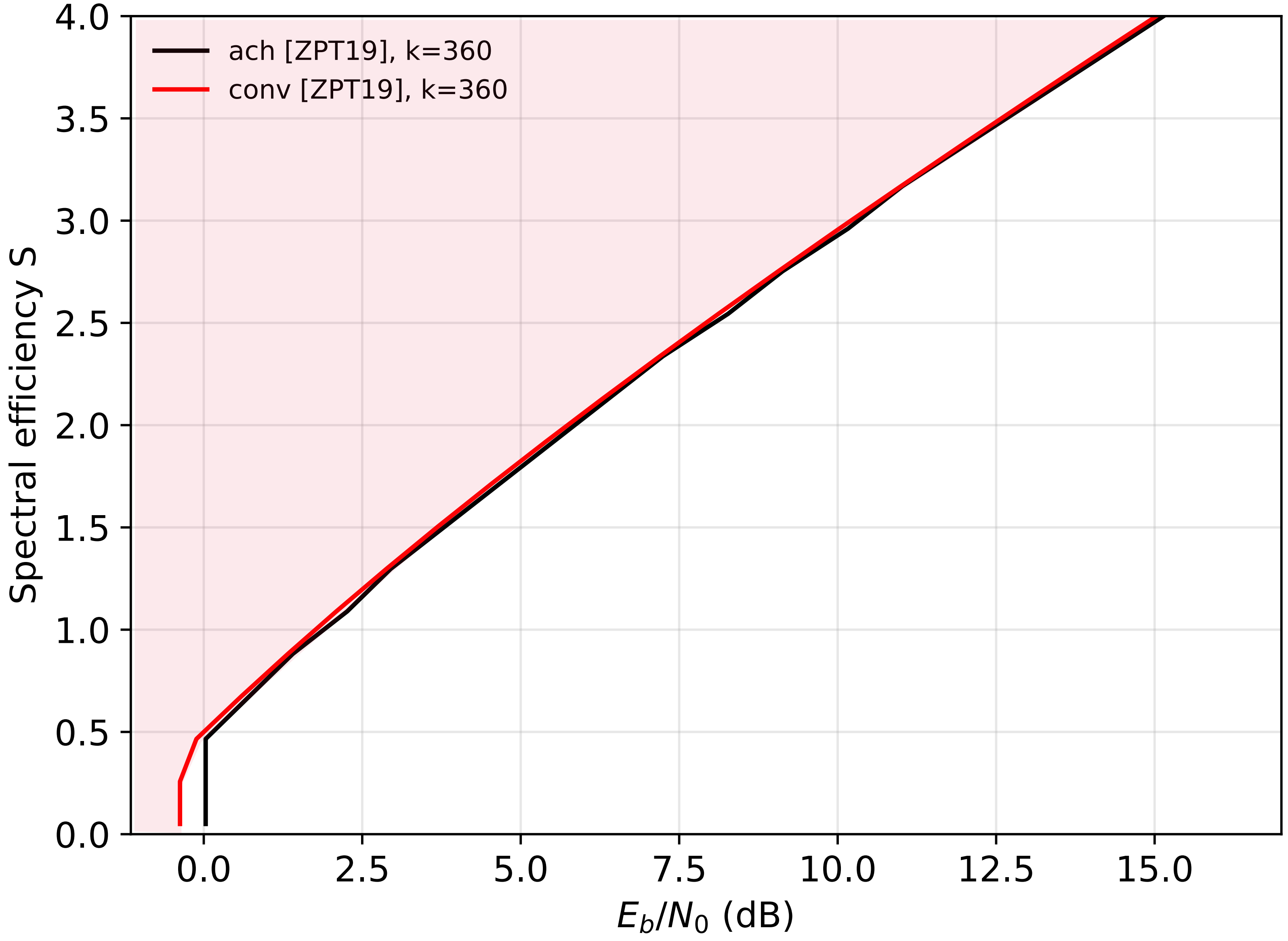
payload 360 bits, target $\texttt{BER} = 10^{-4}$



Spectral efficiency: total # bits / total # channel uses

$$\mathsf{S} = \mu k = Lk/n$$

payload 360 bits, target $\mathtt{BER} = 10^{-4}$



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[Zadik, Polyanskiy, Thrampoulidis '19]

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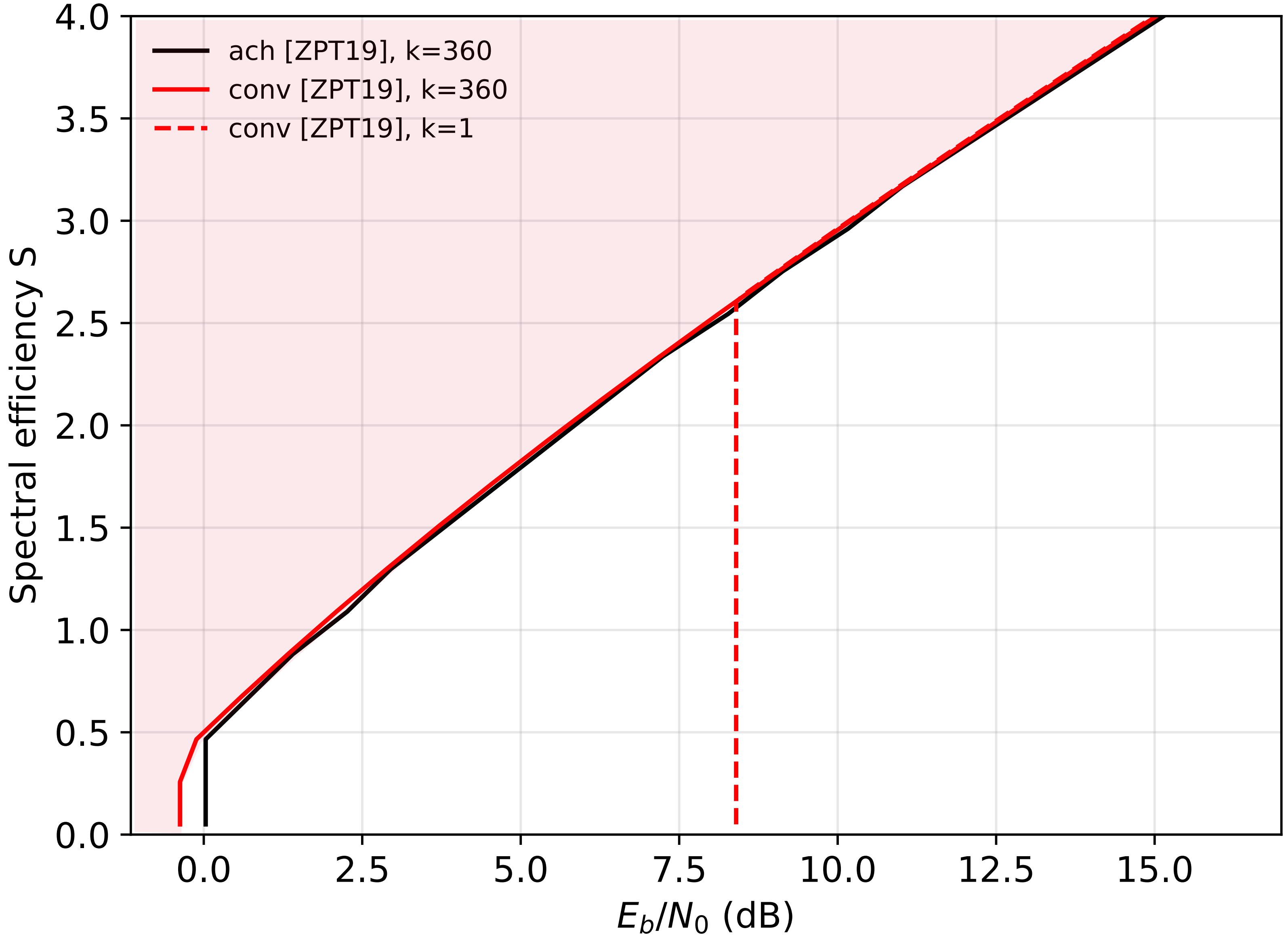

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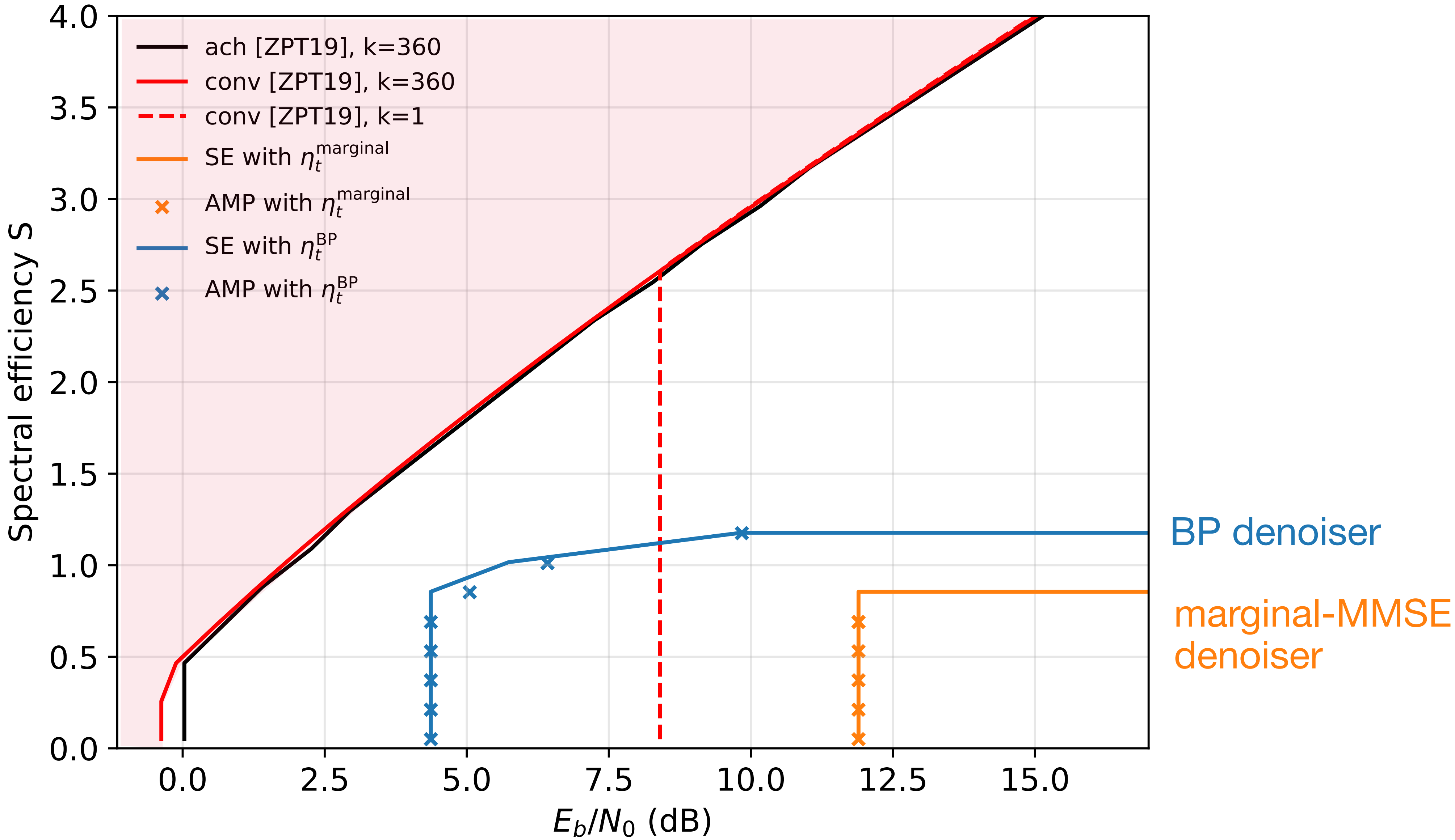
[Zadik, Polyanskiy, Thrampoulidis '19]

Thanks to Jossy for BP implementation!

payload 360 bits, target $\texttt{BER} = 10^{-4}$



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[Liu, Hsieh, Venkataramanan '24]

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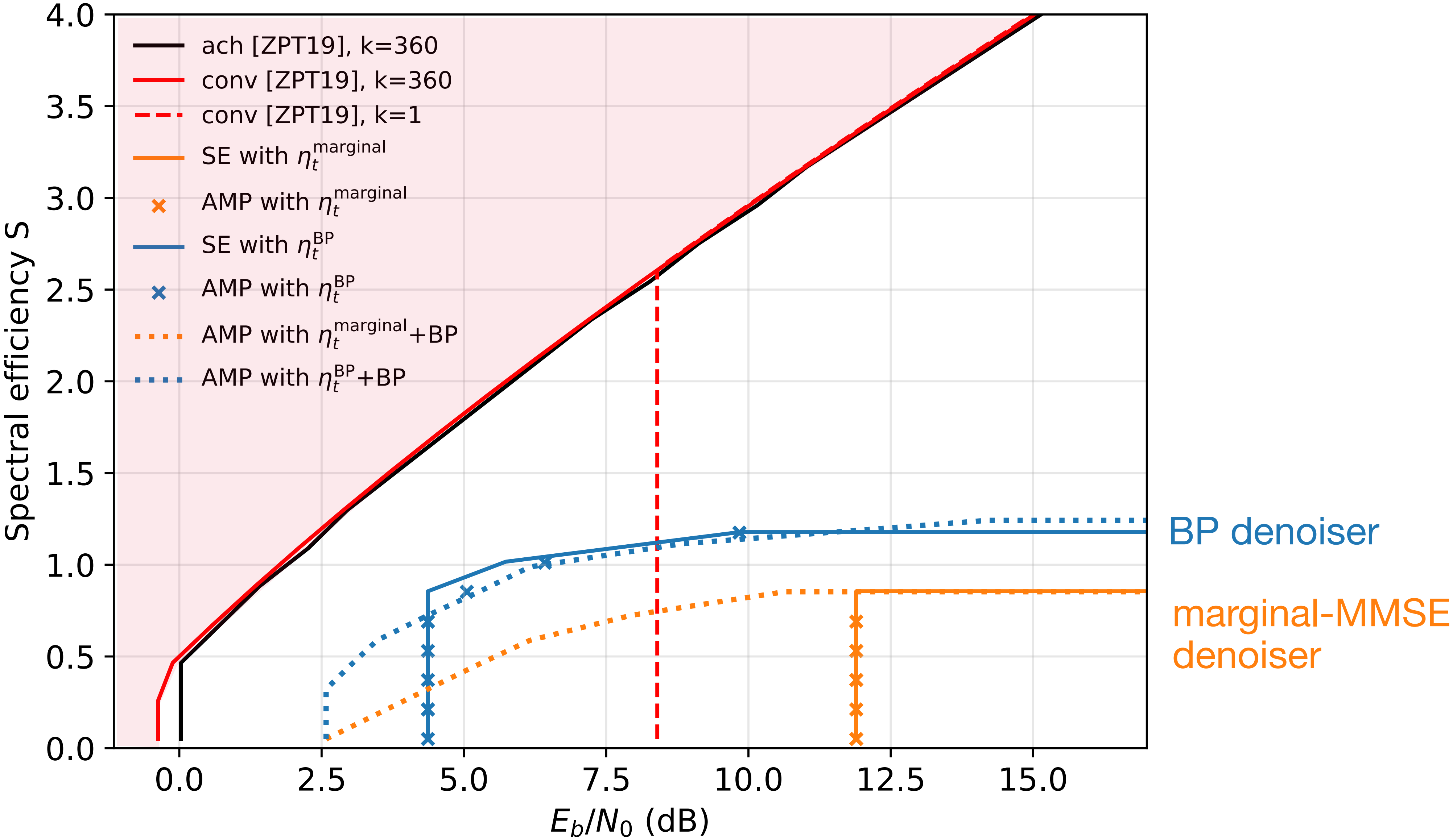

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[Liu, Hsieh, Venkataramanan '24]

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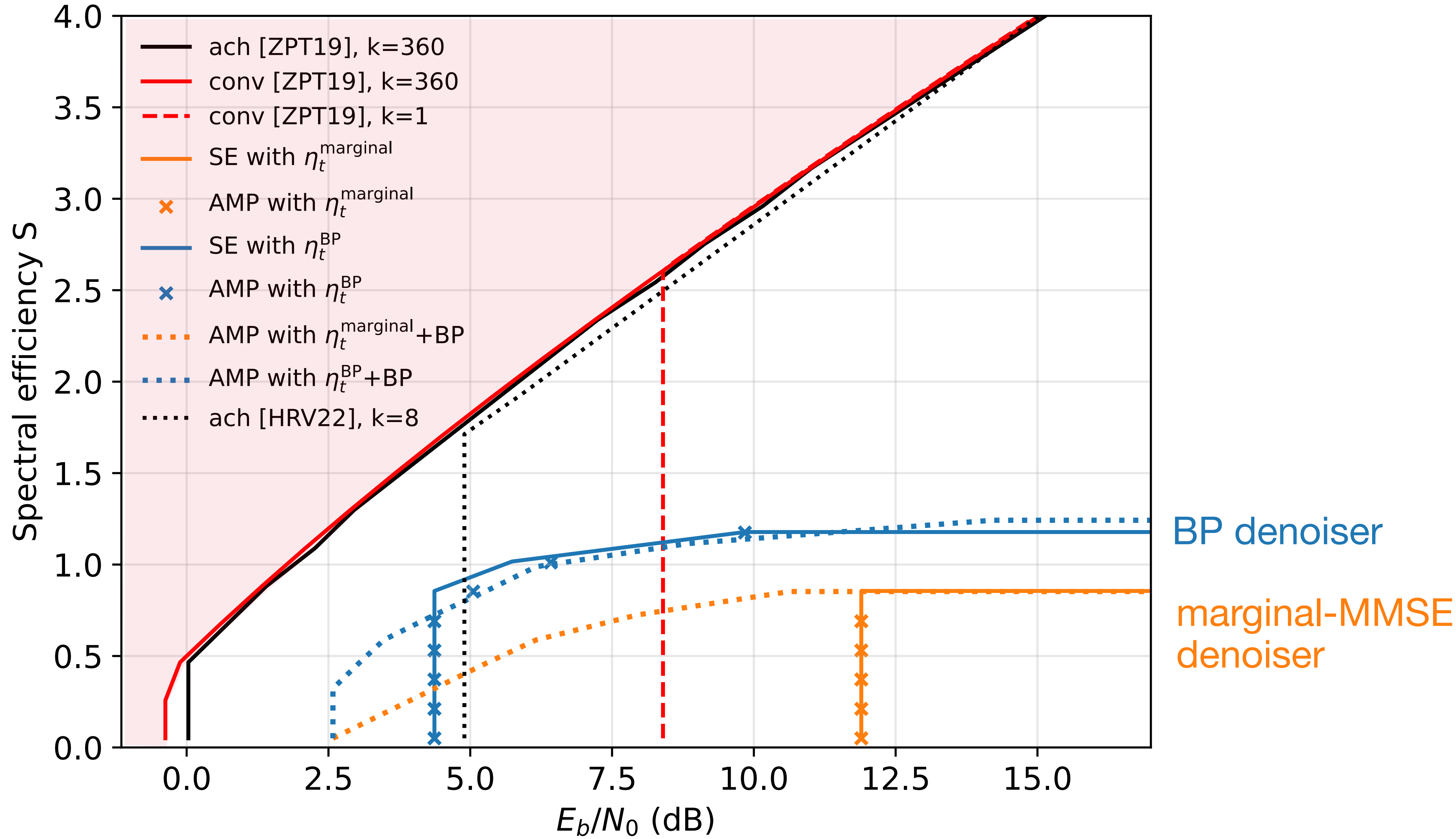

Spectral efficiency: total # bits / total # channel uses

$$\mathsf{S} = \mu k = Lk/n$$

[Liu, Hsieh, Venkataramanan '24]

[Zadik, Polyanskiy, Thrampoulidis '19]

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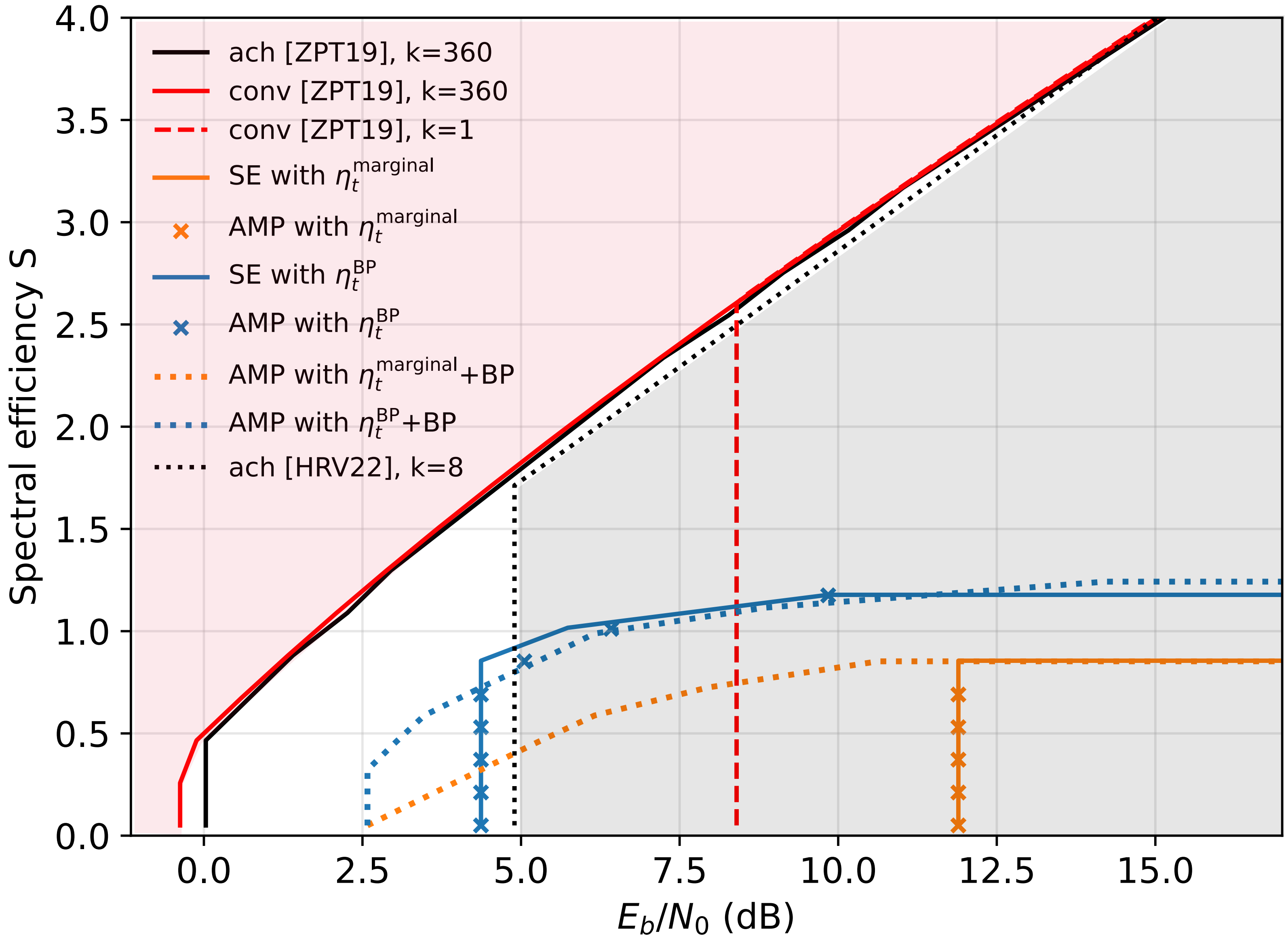


[Hsieh, Rush, Venkataraman '22]

BP denoiser

marginal-MMSE denoiser

Spectral efficiency: total # bits / total # channel uses

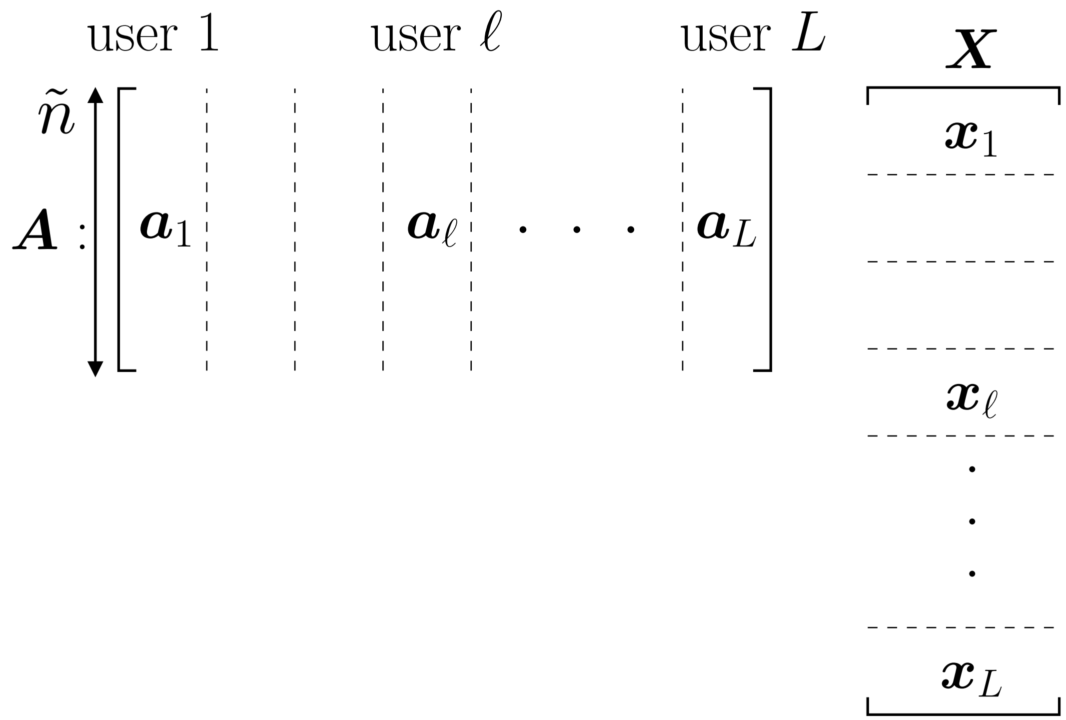
$$\mathsf{S} = \mu k = Lk/n$$

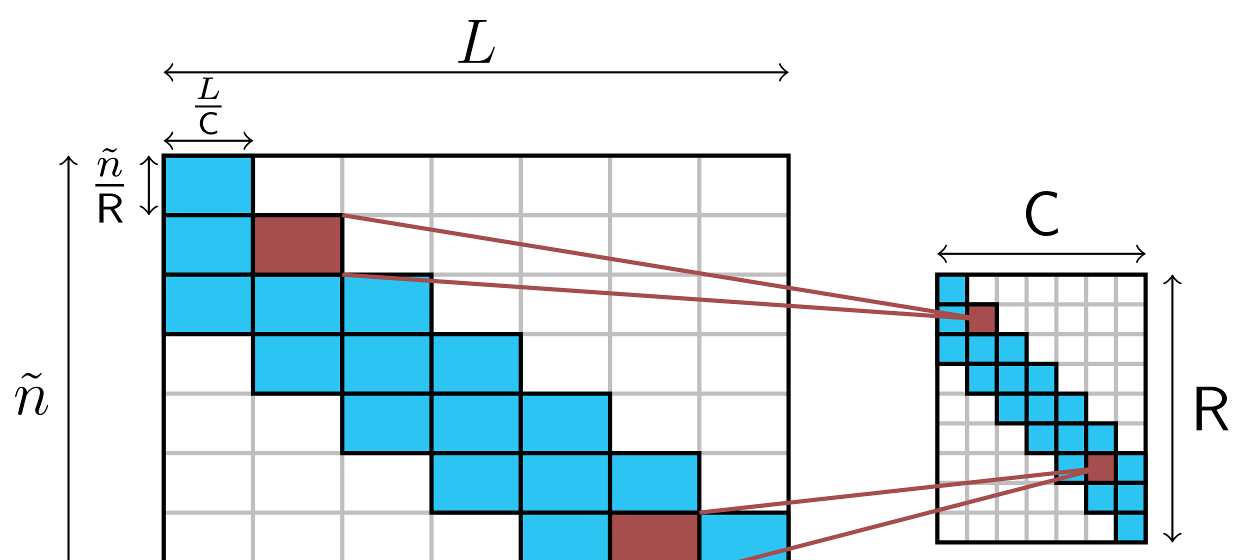
Can we improve at higher S?

[Liu, Hsieh, Venkataraman '24]

[Zadik, Polyanskiy, Thrampoulidis '19]

Spatially coupled (SC) Gaussian matrix $\boldsymbol{A}$





base matrix $\boldsymbol{W}$

combined
codebook matrix $\boldsymbol{A}$

$$A_{i\ell} \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(0, \frac{1}{\tilde{n}/\mathsf{R}} W_{\mathsf{r}(i),\mathsf{c}(\ell)}\right),$$

$$\text{for } i \in [\tilde{n}], \ \ell \in [L]$$

white parts are zeros

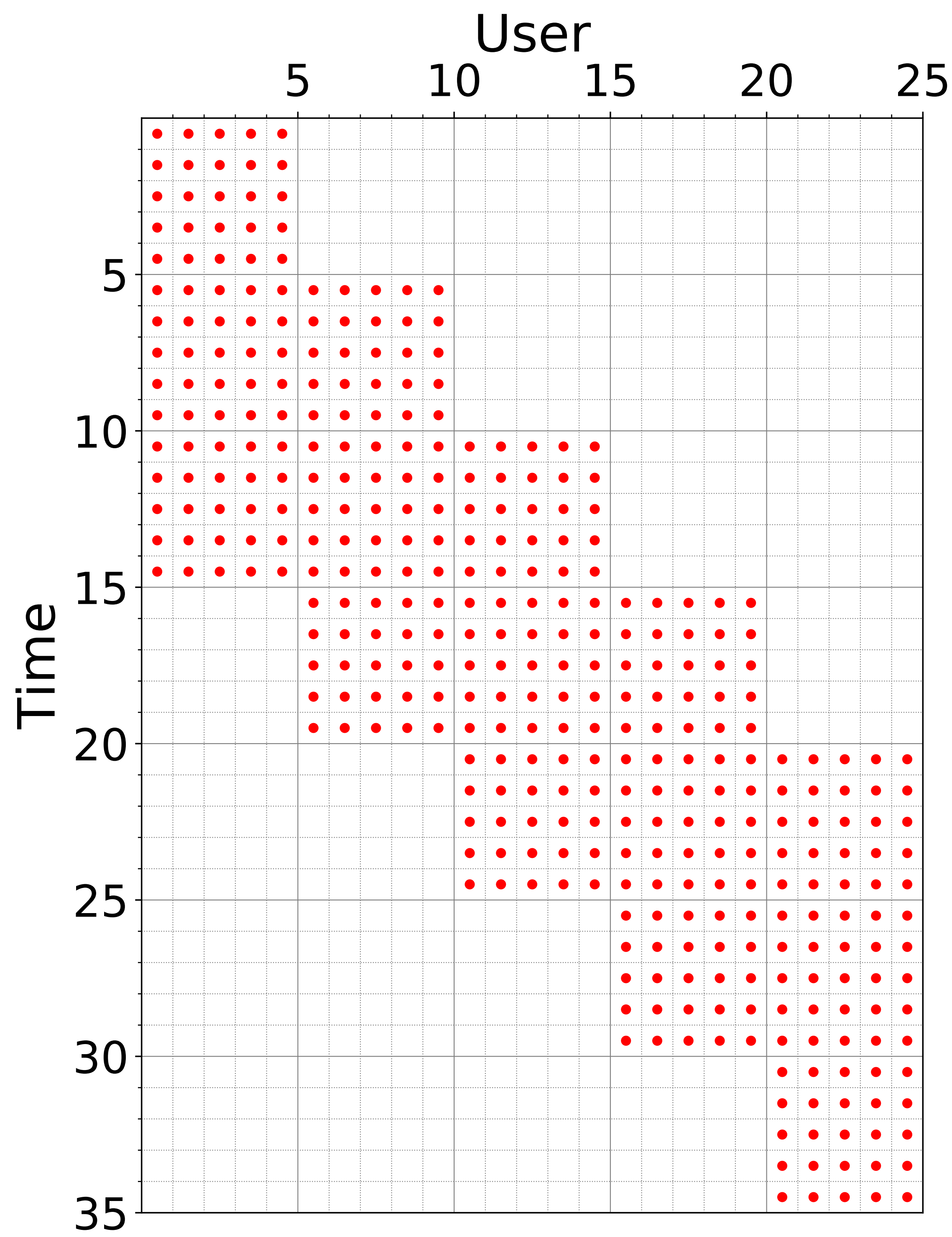
[Kudekar, Pfister '10], [Donoho, Javanmard, Montanari '13] [Barbier and Krzakala '17] [Liang, Ma and Ping '17] [Hsieh, Rush, Venkataramanan '21]…

Spatially coupled (SC) Gaussian matrix A

Example:

$L = 25$ users

$\tilde{n} = 35$ transmissions



Spatially coupled (SC) Gaussian matrix A

Example:

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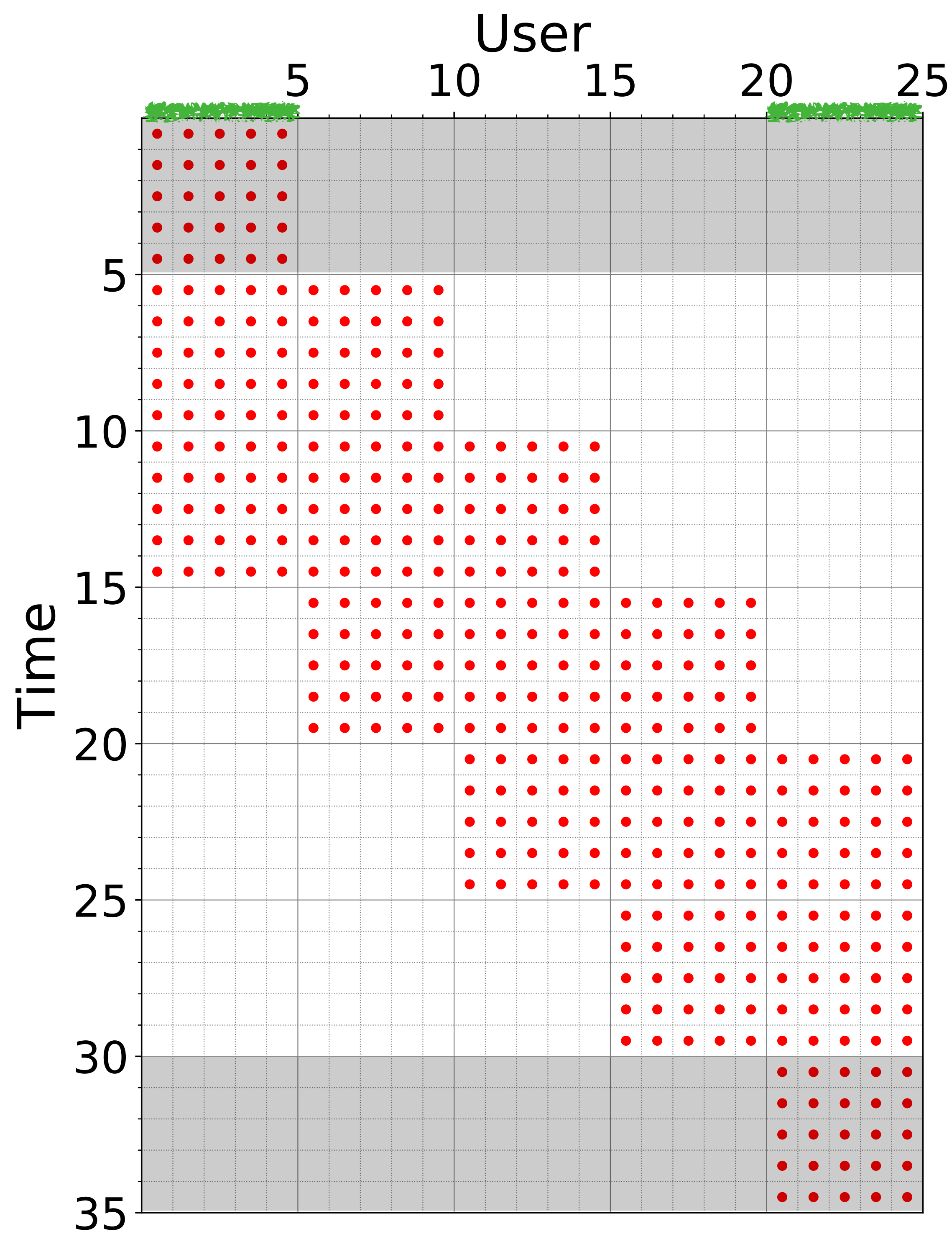
$\tilde{n} = 35$ transmissions



Spatially coupled (SC) Gaussian matrix A

Example:

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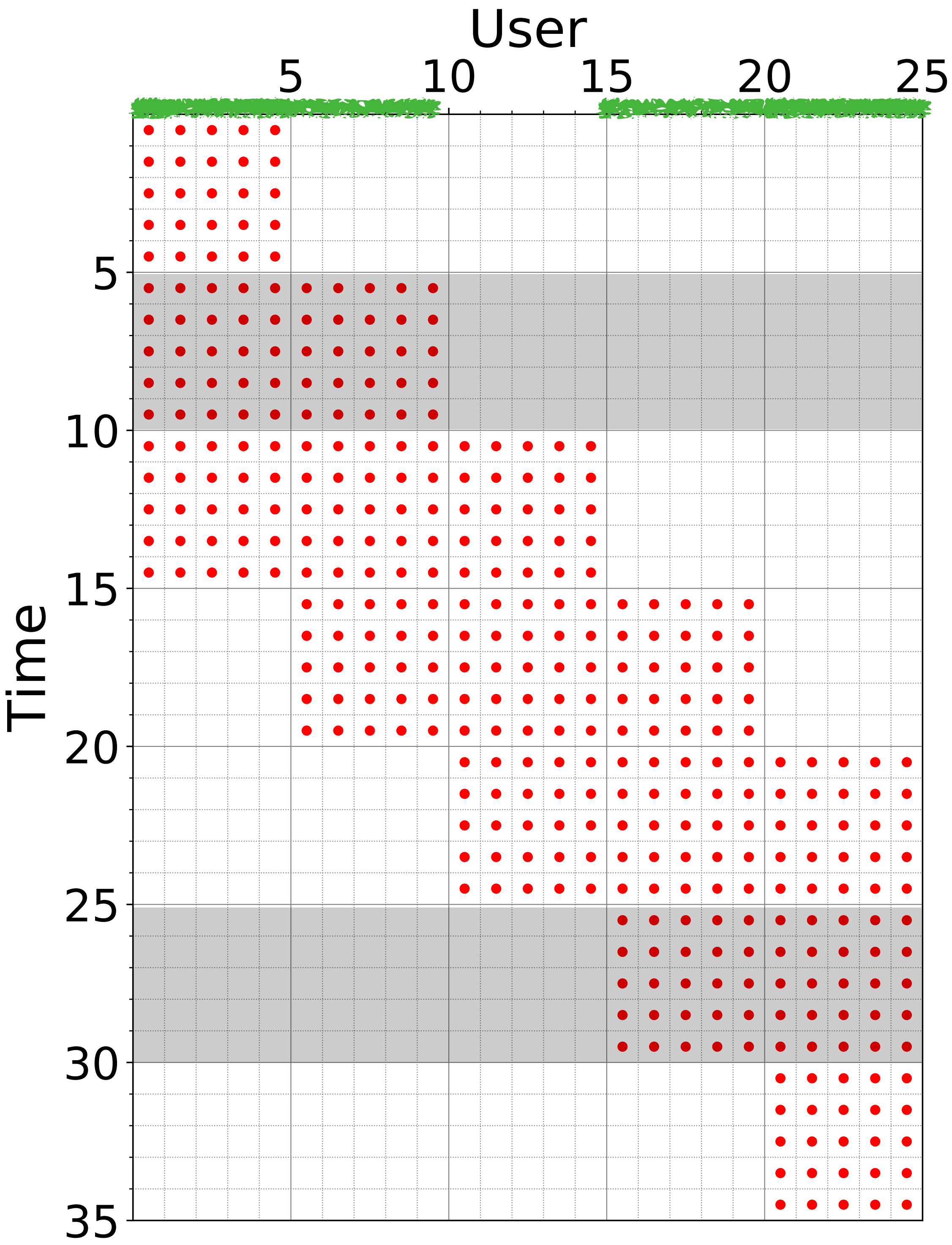

Spatially coupled (SC) Gaussian matrix A

Example:

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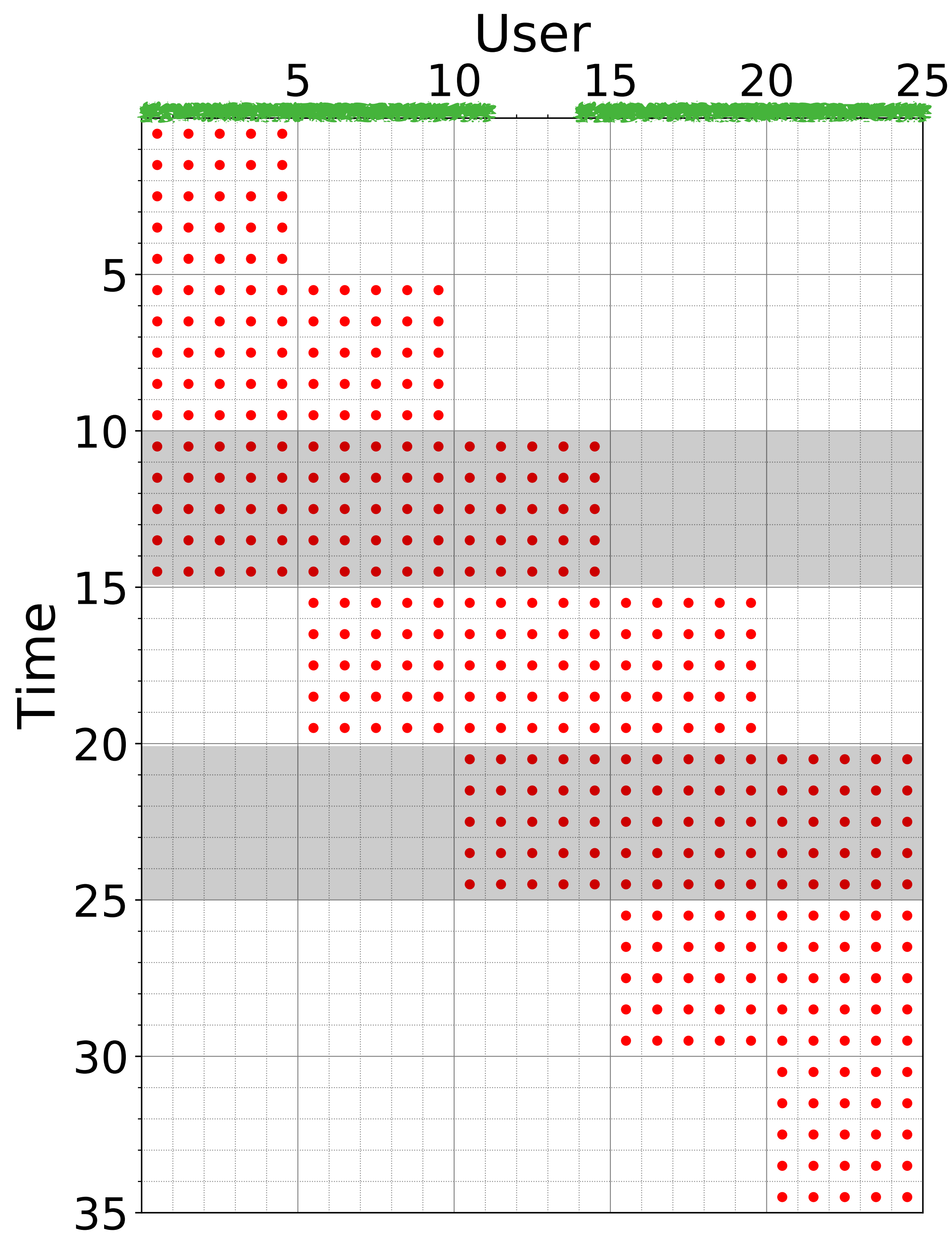
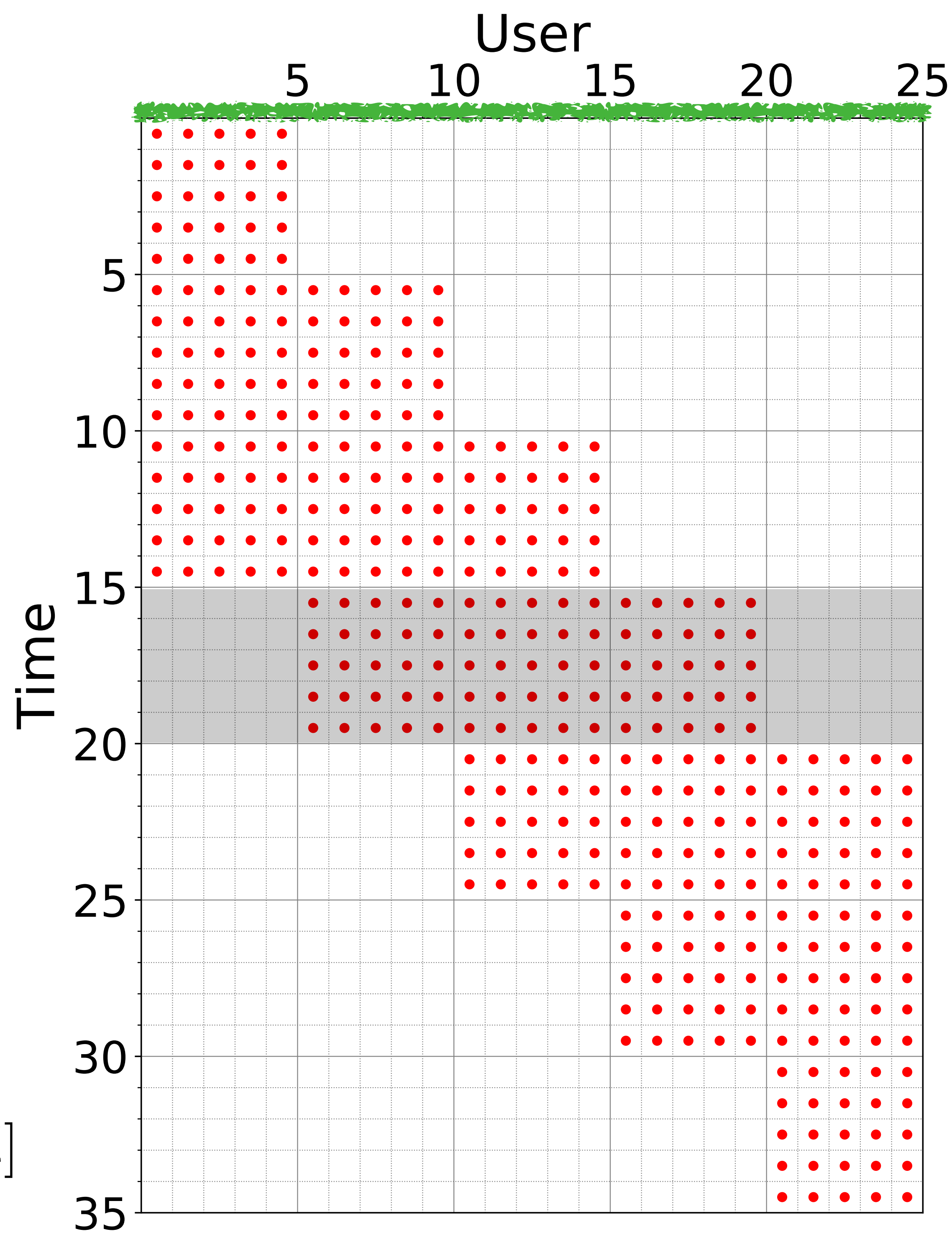
$\tilde{n} = 35$ transmissions



Spatially coupled (SC) Gaussian matrix A



Example:

$L = 25$ users

$\tilde{n} = 35$ transmissions

AMP decoder similar to before, but now with block-dependent terms and denoising functions $\eta_{t,c}$ for $c \in [C]$

Theorem for SC Gaussian $\boldsymbol{A}$ [Liu, Hsieh, Venkataramanan '24]

Let $\eta_{1,\mathsf{c}}, \dots, \eta_{t,\mathsf{c}}$ be Lipschitz for $\mathsf{c} \in [\mathsf{C}]$, then

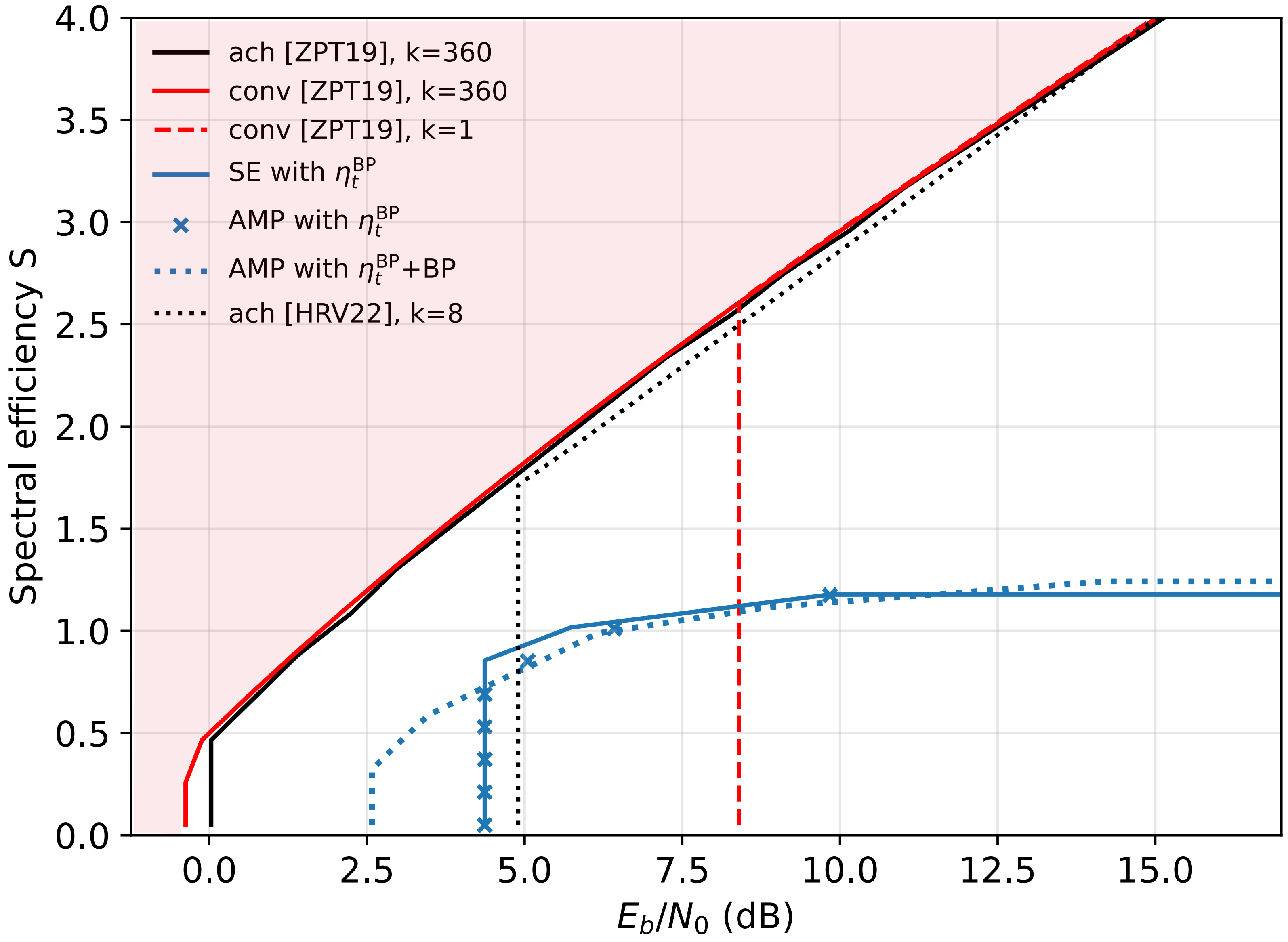
$$\lim_{L\to\infty} \mathtt{UER} := \lim_{L\to\infty} \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{1}\{\hat{\boldsymbol{x}}_\ell^{t+1} \neq \boldsymbol{x}_\ell\} = \boxed{\frac{1}{\mathsf{C}}\sum_{\mathsf{c}=1}^{\mathsf{C}} \mathbb{P}\left(h_{t,\mathsf{c}}\left(\bar{\boldsymbol{x}} + \boldsymbol{g}_{\mathsf{c}}^t\right) \neq \bar{\boldsymbol{x}}\right)}$$

deterministic

$$\lim_{L\to\infty} \mathtt{BER} := \lim_{L\to\infty} \frac{1}{Ld}\sum_{\ell=1}^{L}\sum_{j=1}^{d} \mathbb{1}\left\{\hat{x}_{\ell,j}^{t+1} \neq x_{\ell,j}\right\} = \boxed{\frac{1}{\mathsf{C}}\sum_{\mathsf{c}=1}^{\mathsf{C}} \frac{1}{d}\sum_{j=1}^{d} \mathbb{P}\left(\left[h_{t,\mathsf{c}}\left(\bar{x} + g_{\mathsf{c}}^t\right)\right]_j \neq \bar{x}_j\right)}$$

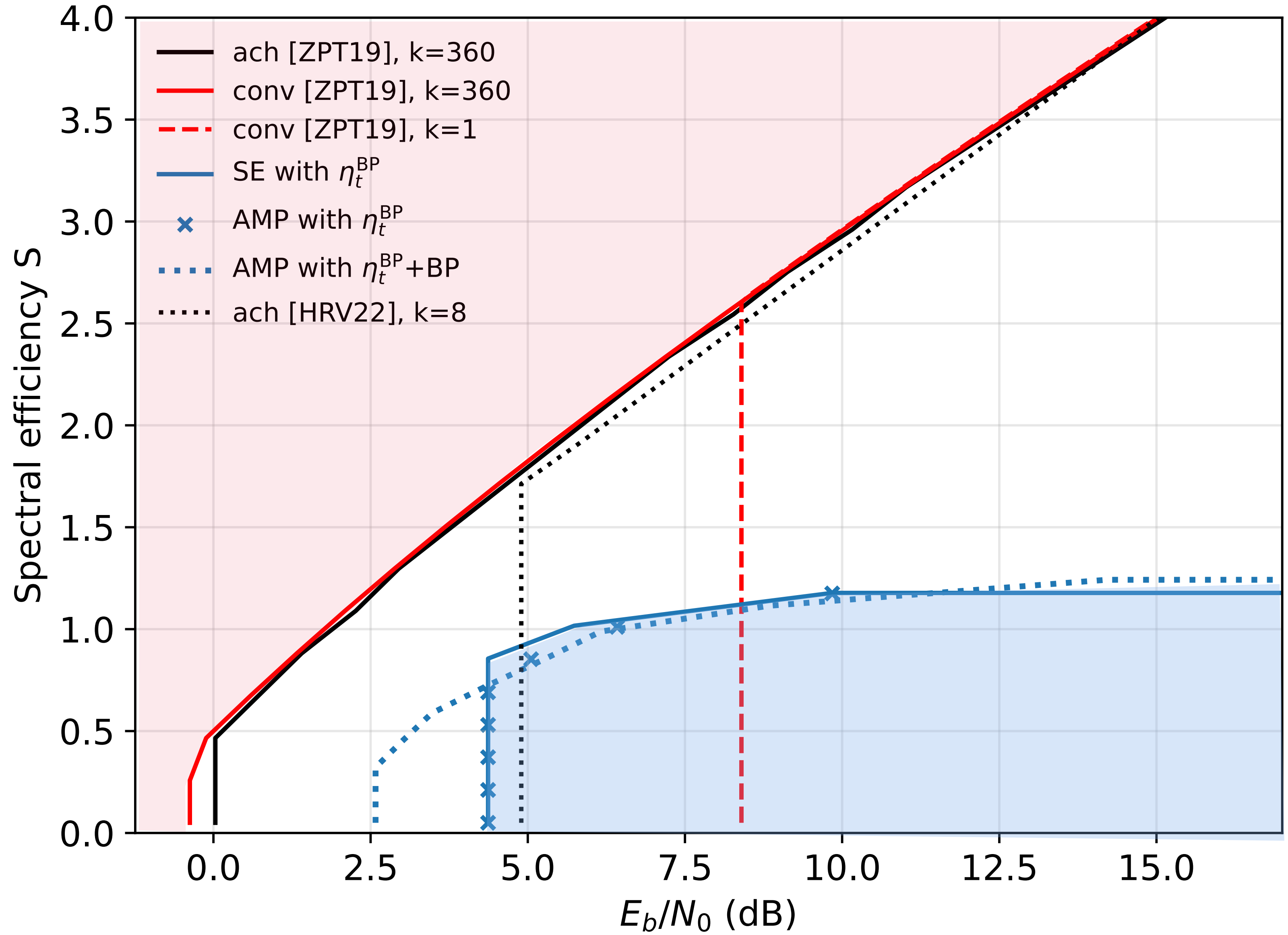
- $\bar{\boldsymbol{x}} \in \{\pm\sqrt{E}\}^d$ uniformly distributed among 2^k codewords
- $\boldsymbol{g}_{\mathsf{c}}^t \in \mathbb{R}^d \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathsf{c}}^t)$ independent of $\bar{\boldsymbol{x}}$

payload 360 bits, target $\mathtt{BER} = 10^{-4}$



iid Gaussian design +
AMP w. BP denoiser

payload 360 bits, target $\mathtt{BER} = 10^{-4}$



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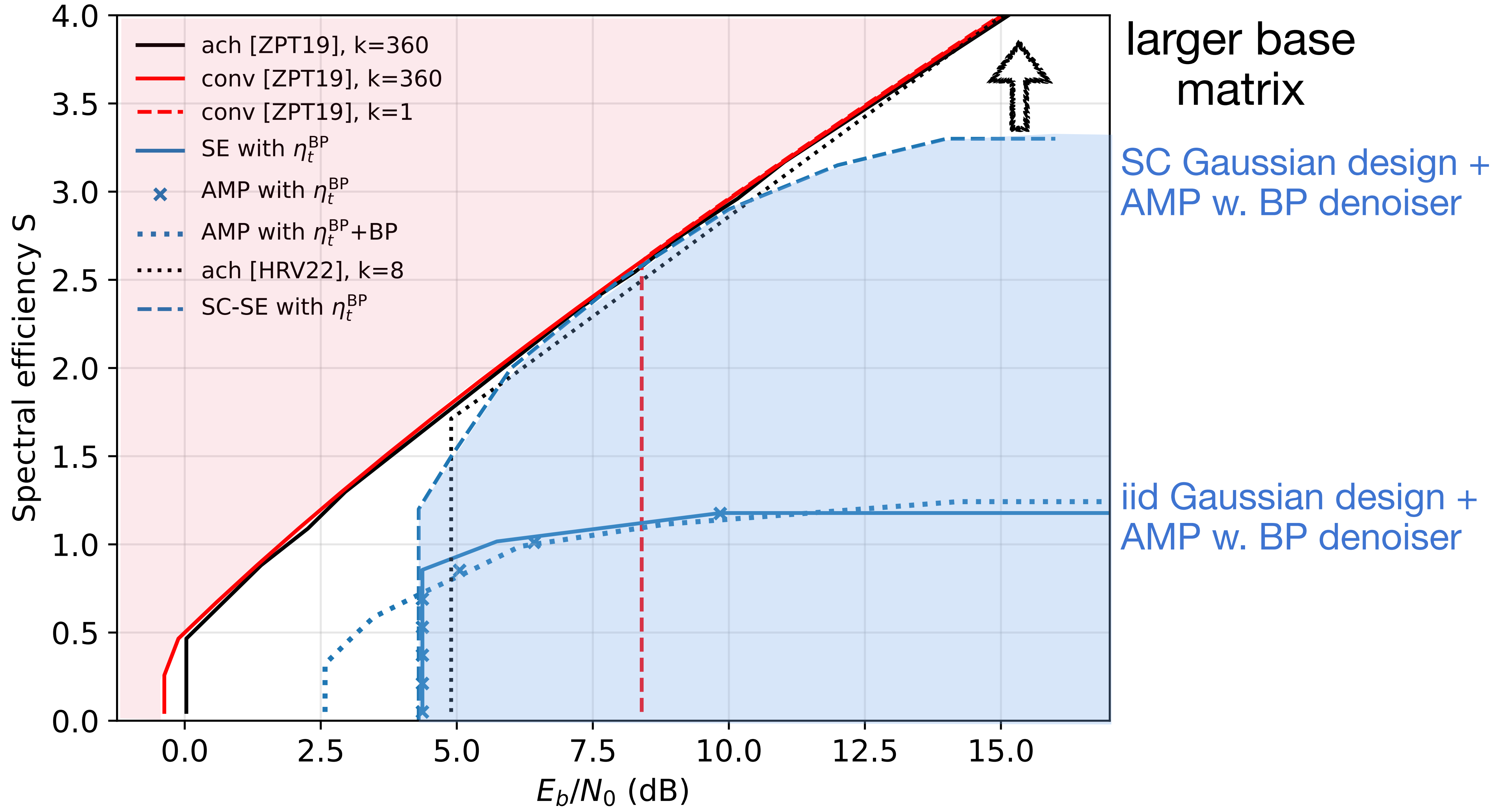


Many-user multiple-access

Previous work [Polyanskiy '17], [Zadik, Polyanskiy, Thrampoulidis '19], [Polyanskiy, Kowshik '20]

What can be achieved without **memory** or **computational** constraints?

random Gaussian codebooks + maximum-likelihood decoding

This talk [Hsieh, Rush, Venkataramanan '22] [Liu, Hsieh, Venkataramanan '24]

What can be achieved with efficient coding schemes?

random linear coding + AMP decoding

(last 5mins of talk)

Many-user multiple-access with random user activity

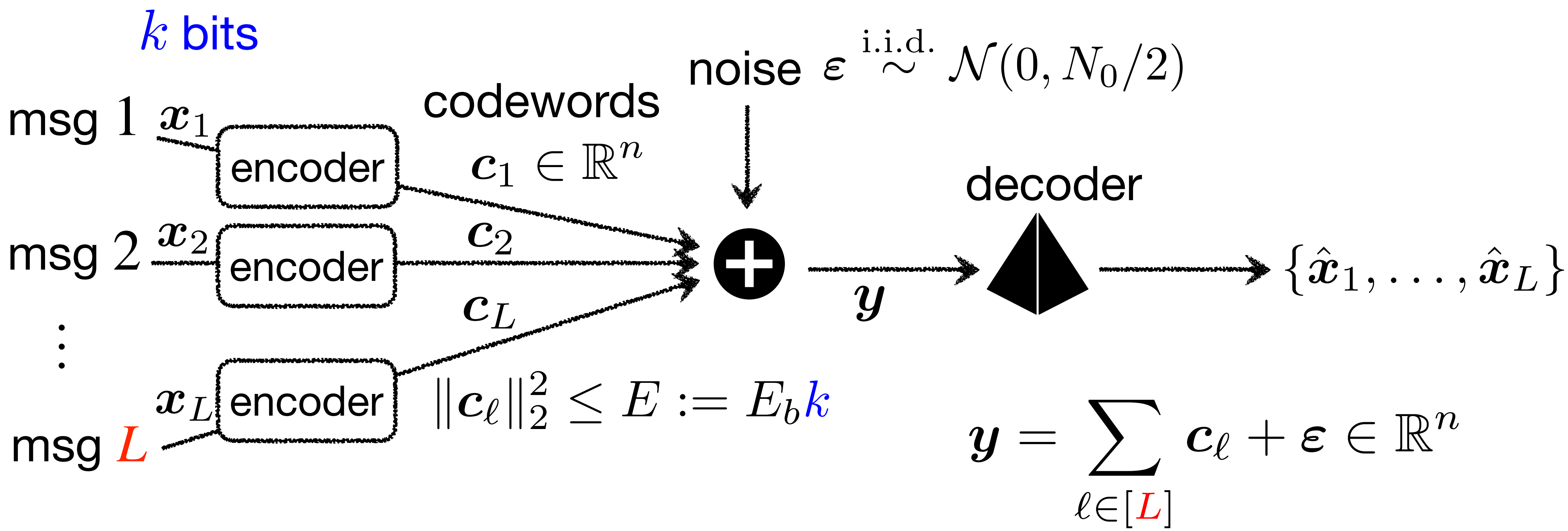
[Chen, Guo '14], [Ordentlich, Polyanskiy '17], [Yavas, Kostina, Effros '21], [Ngo, Lancho, Durisi, Graell i Amat '23]

[Liu, Cobo, Venkataramanan '24]

What can be achieved without **memory** or **computational** constraints?

What can be achieved with efficient coding schemes?

Many-user GMAC with random user activity

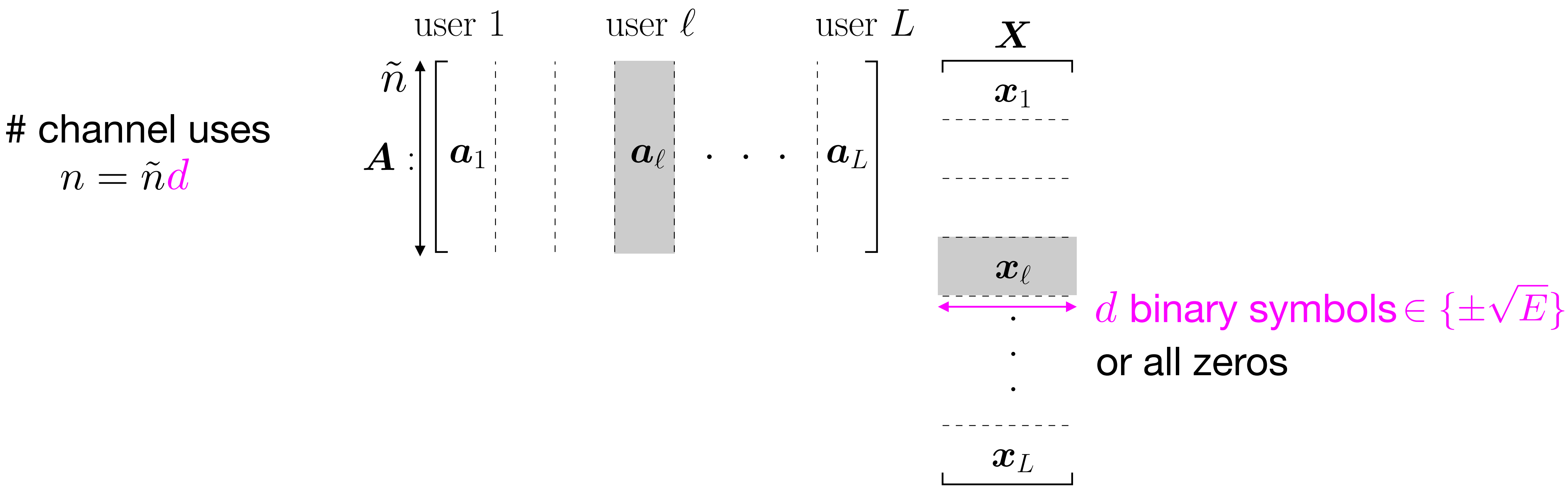


- Each user active with probability α
- Errors: mis-detections (MD), false alarms (FA), active-user errors (AUE)

Linear scaling regime: $L, n \to \infty$ with $\mu := L/n$ fixed

Given $\mu = L/n$, what is minimum E_b/N_0 required to achieve target error rates?

Random binary-CDMA

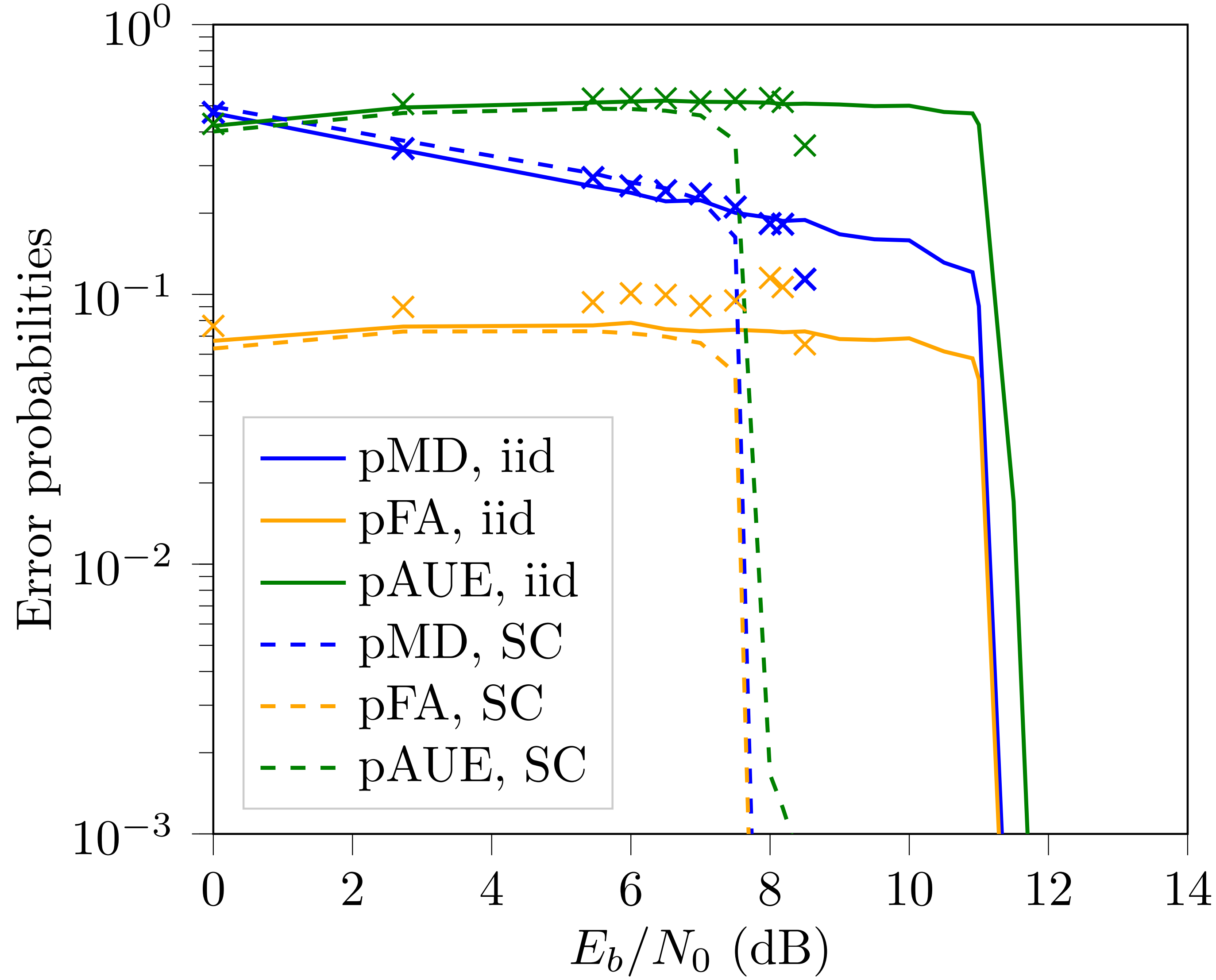


- $\boldsymbol{x}_\ell \in \{\pm\sqrt{E}\}^d \sim p_{\bar{\boldsymbol{x}}}$ with probability α, or all-zero with probability $(1-\alpha)$
- AMP decoder can be adapted to this prior distribution
- Asymptotic $p_{\mathrm{MD}}, p_{\mathrm{FA}}, p_{\mathrm{AUE}}$ can be precisely quantified

[Liu, Cobo, Venkataramanan '24]

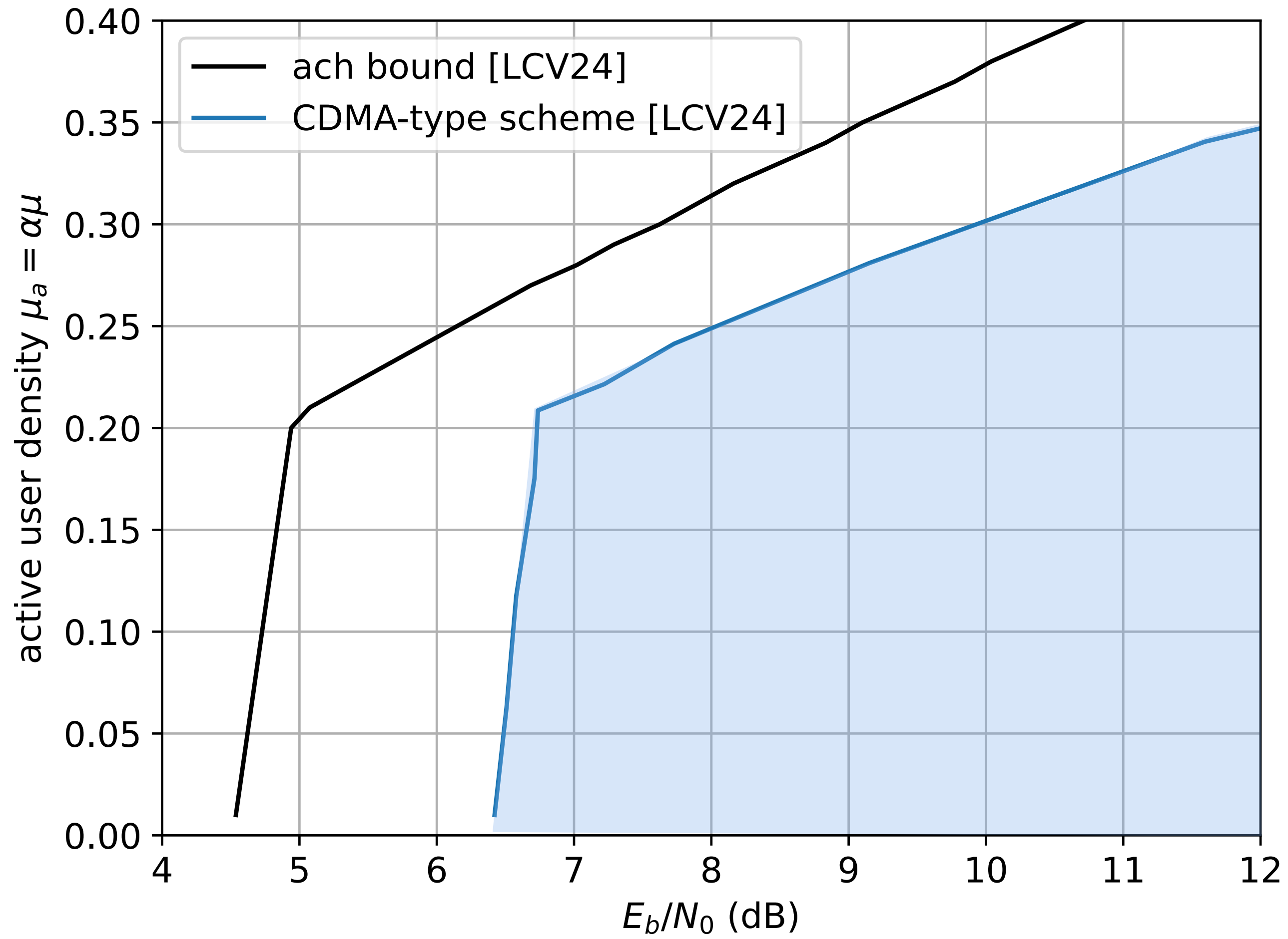
random binary-CDMA + AMP decoding

user payload $k = 8$, $\alpha = 0.7$



user payload $k = 6$, $\alpha = 0.7$

target error $p_{\text{total}} = \max\{p_{\text{MD}}, p_{\text{FA}}\} + p_{\text{AUE}} \leq 0.01$



Asymptotic achievability bounds:

- based on random linear coding + AMP decoding
- scheme impractical for larger k, but its asymptotic performance can be precisely and efficiently characterised

Summary

Many-user Gaussian multiple-access

[Liu, Hsieh, Venkataramanan '24]

- Near-optimal performance for larger payloads k via

  random binary-CDMA with outer code + AMP decoding with BP denoiser

- Memory and computational costs linear in payload k

- Exact asymptotic error guarantees

- Extension to incorporate random user activity [Liu, Cobo, Venkataramanan '24]

X. Liu, K. Hsieh, and R. Venkataramanan, *Coded many-user multiple access via Approximate Message Passing*, https://arxiv.org/abs/2402.05625, 2024

X. Liu, P. Pascual Cobo, and R. Venkataramanan, *Many-user multiple access with random user activity* (to appear at ISIT 2024)