Many-user multiple access with random user activity

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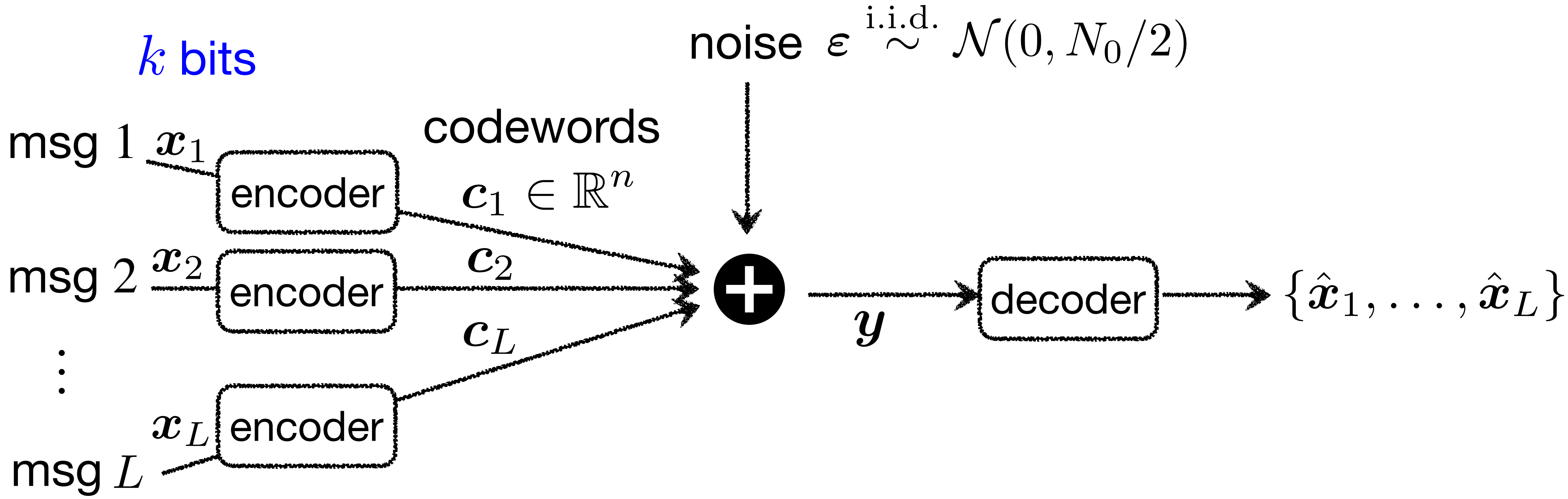
Gaussian multiple-access channel (GMAC)



$$\boldsymbol{y} = \sum_{\ell \in [L]} \boldsymbol{c}_\ell + \boldsymbol{\varepsilon} \in \mathbb{R}^n$$

Many-user multiple access

with random user activity

$$\boldsymbol{y} = \sum_{\ell \in [L]} \boldsymbol{c}_\ell + \boldsymbol{\varepsilon} \in \mathbb{R}^n$$

- each user active w. p. α, K_a active users
- active user density $\mu_\mathrm{a} := \mathbb{E}[K_\mathrm{a}]/n = \alpha L/n$
- fixed user payload k bits
- energy-per-bit constraint: $\|\boldsymbol{c}_\ell\|_2^2 \leq E := E_b k$
- errors: probabilities of mis-detection p_MD, false alarm p_FA, active-user errors p_AUE

Linear scaling regime: $L, n \to \infty$ with α and μ_a fixed.

Given μ_a, what is minimum E_b/N_0 required to achieve target error rates?

[Chen, Guo '14], [Ordentlich, Polyanskiy '17], [Yavas, Kostina, Effros '21], [Ngo, Lancho, Durisi, Graell i Amat '23]

[Liu, Pascual Cobo, Venkataramanan '24]

What can be achieved without memory or computational constraints?

What can be achieved with efficient coding schemes?

Finite-length achievability bounds

Random coding

- Each user $\ell \in [L]$ has $M = 2^k$ codewords $\left\{\boldsymbol{c}_1^{(\ell)}, \boldsymbol{c}_2^{(\ell)}, \dots, \boldsymbol{c}_M^{(\ell)}\right\}$ (iid Gaussian)
- $\mathcal{W}$ denotes set of transmitted codewords, with size $K_{\mathrm{a}} = |\mathcal{W}|$

Maximum-likelihood (ML) decoding

- ML estimate of #active users: $$K'_{\mathrm{a}} = \underset{K_{\mathrm{a}} \in [\kappa_\ell : \kappa_{\mathrm{u}}]}{\arg\max}\ p(\boldsymbol{y}|K_{\mathrm{a}})$$
- Search for ML estimate of $\mathcal{W}$ given size in $[K'_{\mathrm{a}} - r, K'_{\mathrm{a}} + r]$, $r \geq 0$
- $\widehat{\mathcal{W}}$ denotes decoded set of codewords, with size $\widehat{K_{\mathrm{a}}} = |\widehat{\mathcal{W}}|$

[Ngo, Lancho, Durisi, Graell i Amat '23] mis-detection $$p_{\mathrm{MD}} := \mathbb{E}\left[\mathbb{1}\{K_{\mathrm{a}} \neq 0\} \cdot \frac{1}{K_{\mathrm{a}}} \sum_{\ell:(\ell, w_\ell)\in\mathcal{W}} \mathbb{1}\{\widehat{w_\ell} = \emptyset\}\right]$$

false alarm $$p_{\mathrm{FA}} := \mathbb{E}\left[\mathbb{1}\{\widehat{K_{\mathrm{a}}} \neq 0\} \cdot \frac{1}{\widehat{K_{\mathrm{a}}}} \sum_{\ell:(\ell, \widehat{w_\ell})\in\widehat{\mathcal{W}}} \mathbb{1}\{w_\ell = \emptyset\}\right]$$

active user errors $$p_{\mathrm{AUE}} := \mathbb{E}\left[\mathbb{1}\{K_{\mathrm{a}} \neq 0\} \cdot \frac{1}{K_{\mathrm{a}}} \sum_{\ell:(\ell, w_\ell)\in\mathcal{W}} \mathbb{1}\{\widehat{w_\ell} \neq w_\ell\}\right]$$

[Ngo, Lancho, Durisi, Graell i Amat '23]

Theorem 1 (informal) [Liu, Pascual Cobo, Venkataramanan '24]

Consider iid Gaussian codebooks that satisfy the energy-per-bit constraint, and the maximum-likelihood decoder with decoding radius $r \geq 0$. Given L, n, k, α, and for fixed E_b/N_0, we have

$$p_{\mathrm{MD}} \leq \varepsilon_{\mathrm{MD}}\left(L, n, k, \alpha, \frac{E_b}{N_0}, r\right) \quad \text{where}$$

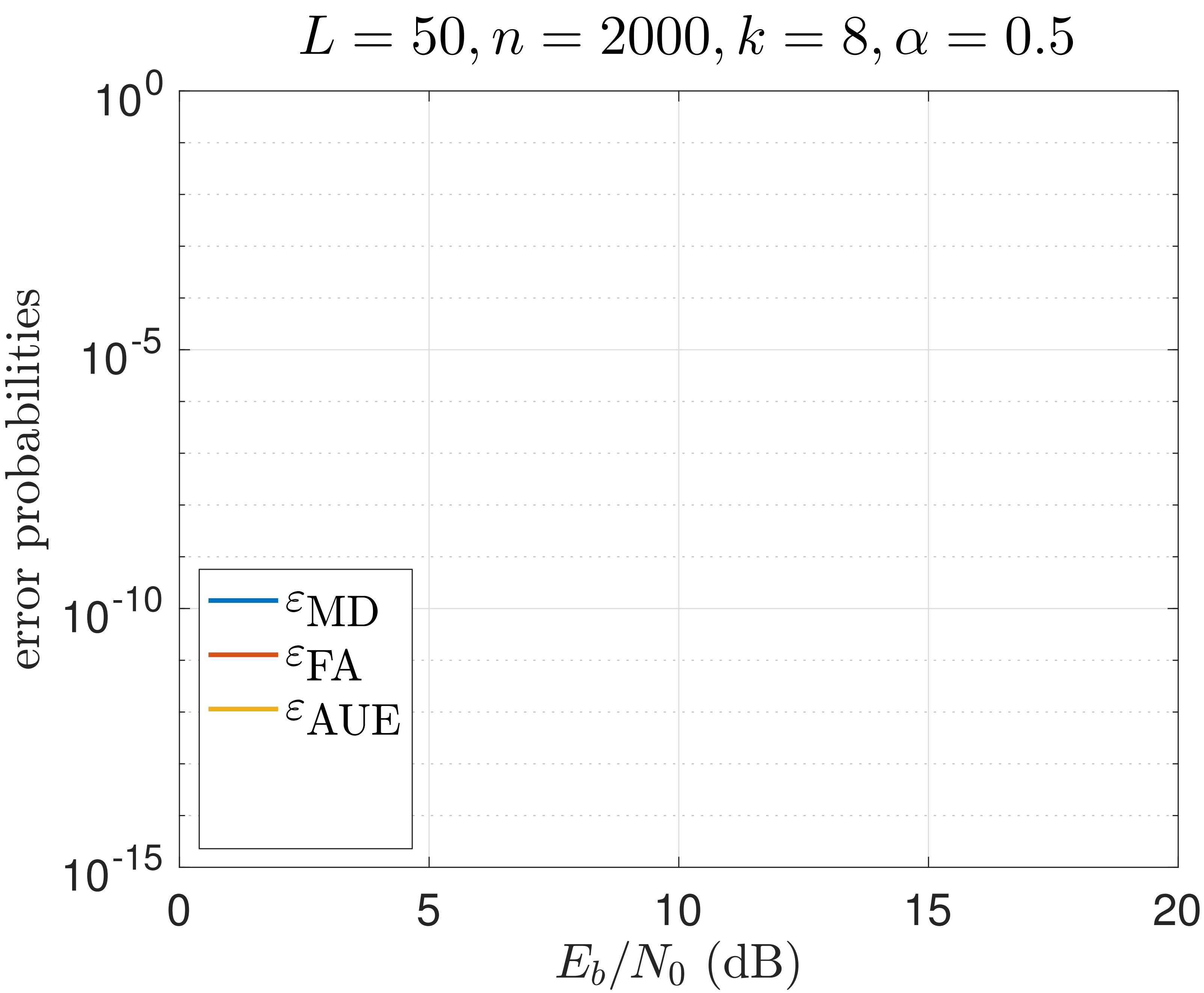
$$\varepsilon_{\mathrm{MD}} = \sum_{\kappa_{\mathrm{a}}} p(\kappa_{\mathrm{a}}; L, \alpha) \sum_{\kappa'_{\mathrm{a}}} \sum_{t \in \mathcal{T}} \sum_{\hat{t} \in \widehat{\mathcal{T}}_t} \min\{p_{t,\hat{t}}, \xi(\kappa_{\mathrm{a}}, \kappa'_{\mathrm{a}})\}$$

$$\cdot \sum_{\psi=0}^{\psi_{\mathrm{u}}} \frac{\nu(t_{\min}, \psi)}{\nu(t_{\min})} \frac{(\kappa_{\mathrm{a}} - \overline{\kappa'_{\mathrm{a}}})^+ + (t - \hat{t})^+ + \psi}{\kappa_{\mathrm{a}}} + \tilde{p}$$

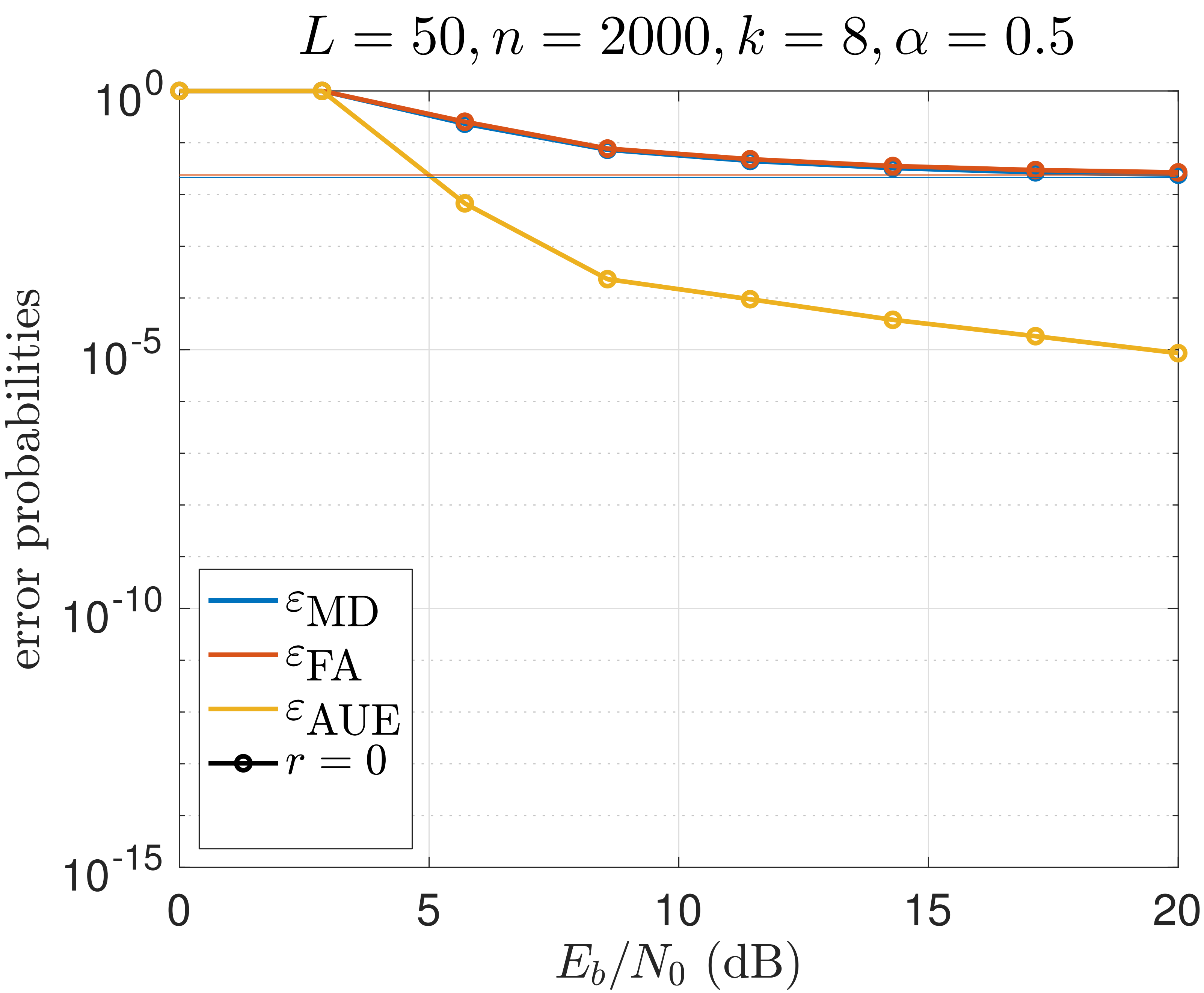
$$p_{\mathrm{FA}} \leq \varepsilon_{\mathrm{FA}}\left(L, n, k, \alpha, \frac{E_b}{N_0}, r\right) \qquad p_{\mathrm{AUE}} \leq \varepsilon_{\mathrm{AUE}}\left(L, n, k, \alpha, \frac{E_b}{N_0}, r\right)$$

Paper provides general version for any distribution of K_{a}

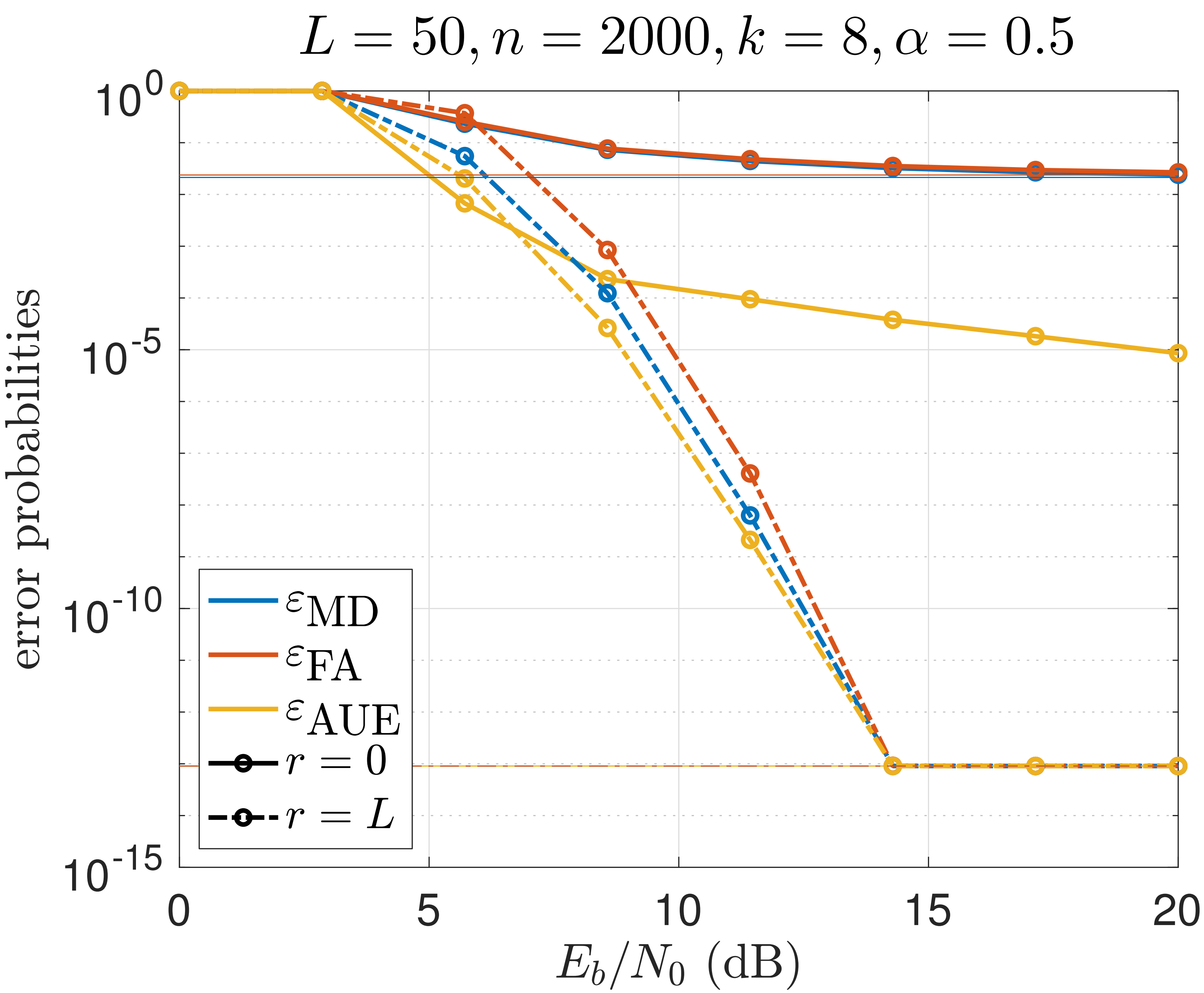
Finite-length achievability bounds



Finite-length achievability bounds



Finite-length achievability bounds



Computationally prohibitive for moderately large L, n

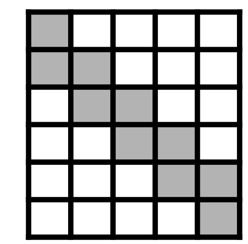
Asymptotic achievability bounds

Random coding

$M = 2^k$ columns

$$\begin{array}{c|c|c|c|c|c} & \text{user } 1 & \text{user } 2 & & \text{user } L & \\ \boldsymbol{A}: & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots & \boldsymbol{A}_L & n \\ \boldsymbol{x}: & \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_L & {}^\top \end{array}$$

spatially-coupled Gaussian $\boldsymbol{A}$



- Channel output: $\boldsymbol{y} = \sum_{\ell \in [L]} \boldsymbol{A}_\ell \boldsymbol{x}_\ell + \boldsymbol{\varepsilon} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}$
- $\boldsymbol{x}_\ell \sim p_{\bar{\boldsymbol{x}}}$ one-sparse with prob. α or all-zero with prob. $1 - \alpha$

Approximate Message Passing (AMP) decoding

Initialise $\boldsymbol{x}^0 = \mathbf{0}$, for $t \geq 1$

Modified residual $\quad \boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \boldsymbol{v} \odot \boldsymbol{z}^{t-1}$

Updated estimate $\quad \boldsymbol{x}^{t+1} = \eta_t(\boldsymbol{s}^t),$ where $\boldsymbol{s}^t = \boldsymbol{x}^t + (\boldsymbol{Q}^t \odot \boldsymbol{A})^\top \boldsymbol{z}^t$

$$
\begin{array}{ccccc}
 & \text{user } 1 & \text{user } 2 & & \text{user } L \\
 & \overleftrightarrow{\quad M = 2^k \text{ columns} \quad} & & & \\
\boldsymbol{A} : & \left[\boldsymbol{A}_1 \right. & \boldsymbol{A}_2 & \cdots & \left. \boldsymbol{A}_L \right] \updownarrow n \\
\boldsymbol{x} : & \left[\boldsymbol{x}_1 \right. & \boldsymbol{x}_2 & \cdots & \left. \boldsymbol{x}_L \right]^{\top}
\end{array}
$$

Asymptotic fixed-point of AMP

Can be characterised via single-user channel

$$
\boldsymbol{s}_{\tau^*} = \underset{\bar{\boldsymbol{x}} \sim p_{\bar{\boldsymbol{x}}}}{\underset{\uparrow}{\bar{\boldsymbol{x}}}} + \sqrt{\tau^*}\underset{\boldsymbol{g} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_B)}{\underset{\nwarrow}{\boldsymbol{g}}} \in \mathbb{R}^M
$$

Where τ^* is the global minimiser of potential function:

$$
\mathcal{F}(\tau) = I(\bar{\boldsymbol{x}}; \boldsymbol{s}_\tau) + \frac{1}{2\mu}\left[\ln\left(\frac{\tau}{N_0/2}\right) - \left(1 - \frac{N_0/2}{\tau}\right)\right]
$$

with $\tau \in \left[\frac{N_0}{2}, \frac{N_0}{2} + \mu E\right]$.

[Barbier, Krzakala '14], [Yedla, Jian, Nguyen, Pfister '14], [Hsieh, Rush, Venkataramanan '22], [Kowshik '22]

Theorem 2 (informal) [Liu, Pascual Cobo, Venkataramanan '24]

Consider spatially coupled Gaussian $\boldsymbol{A}$, message vectors $\boldsymbol{x}_\ell \sim p_{\bar{\boldsymbol{x}}}$ and AMP decoding. For any $\delta > 0$, we have

$$\lim_{t\to\infty} \lim_{L\to\infty} p_{\mathrm{MD}} \leq \varepsilon_{\mathrm{MD}}(\tau^* + \delta) \quad \text{where}$$

$$\varepsilon_{\mathrm{MD}}(\tau) := \Phi\left(\frac{\ln\left[\frac{B}{\alpha}(1-\alpha)\right]}{\sqrt{\frac{E}{\tau}}} - \frac{1}{2}\sqrt{\frac{E}{\tau}}\right)\Phi\left(\frac{\ln\left[\frac{B}{\alpha}(1-\alpha)\right]}{\sqrt{\frac{E}{\tau}}} + \frac{1}{2}\sqrt{\frac{E}{\tau}}\right)^{B-1}$$

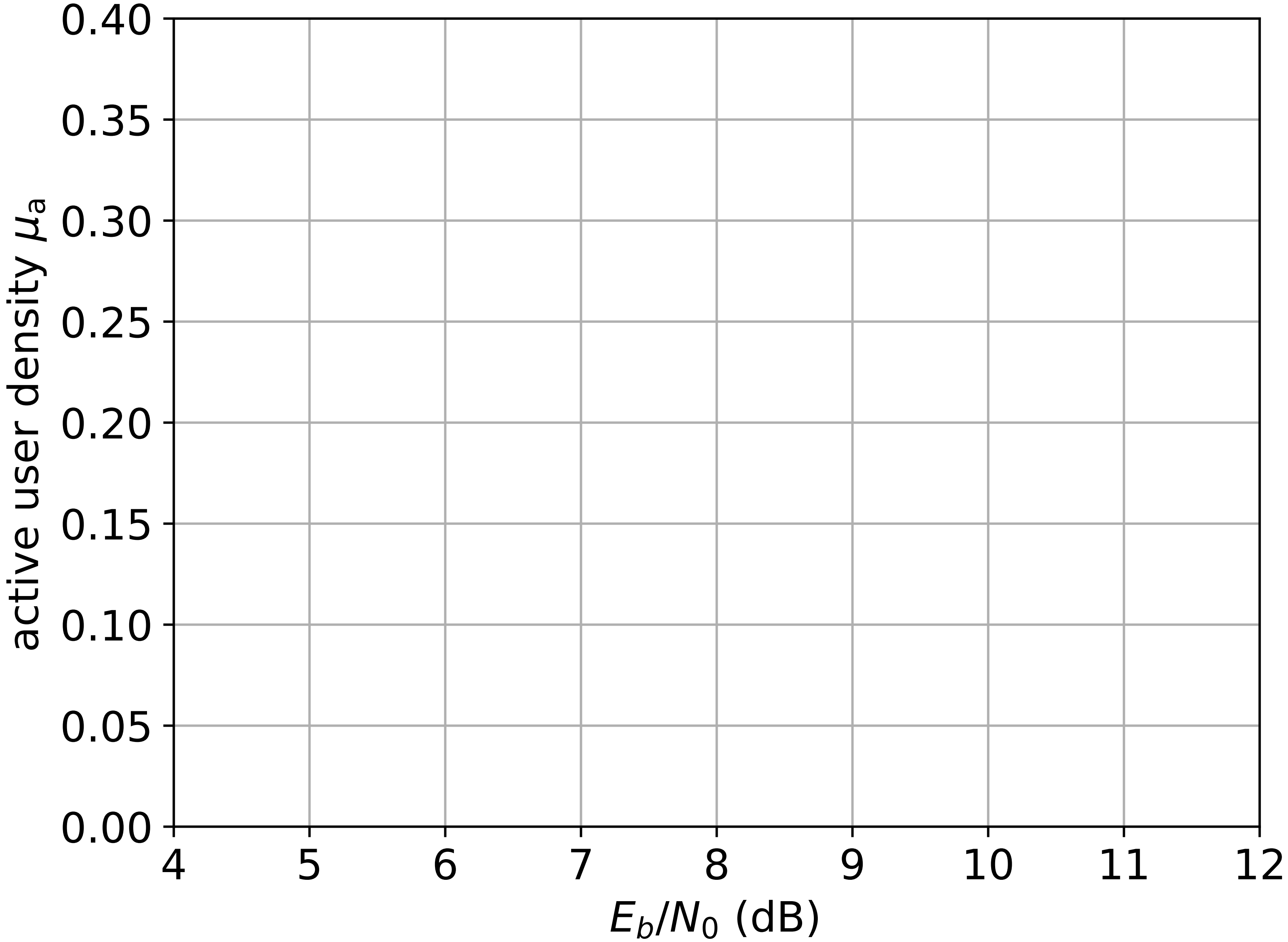
$$\lim_{t\to\infty} \lim_{L\to\infty} p_{\mathrm{FA}} \leq \varepsilon_{\mathrm{FA}}(\tau^* + \delta) \qquad \lim_{t\to\infty} \lim_{L\to\infty} p_{\mathrm{AUE}} \leq \varepsilon_{\mathrm{AUE}}(\tau^* + \delta)$$

Inner limit: $L, n \to \infty$ with $L/n \to \mu$

- Easier to compute than the finite-length bounds
- Can handle large payloads

$k = 6,\ \alpha = 0.7,$

target total error $\max\{p_{\mathrm{MD}}, p_{\mathrm{FA}}\} + p_{\mathrm{AUE}} \leq 0.01$



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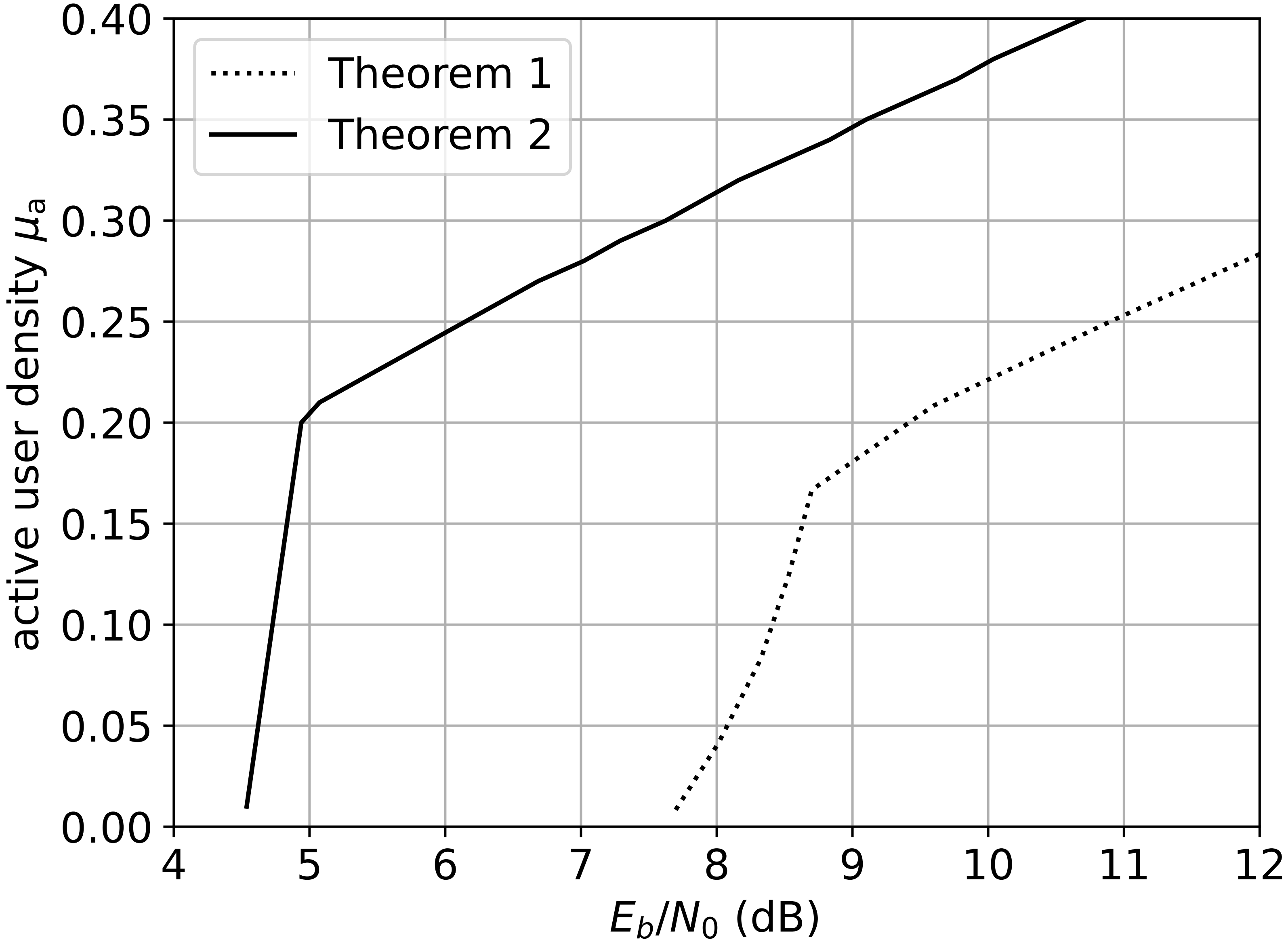

[Liu, Pascual Cobo, Venkataraman '24]

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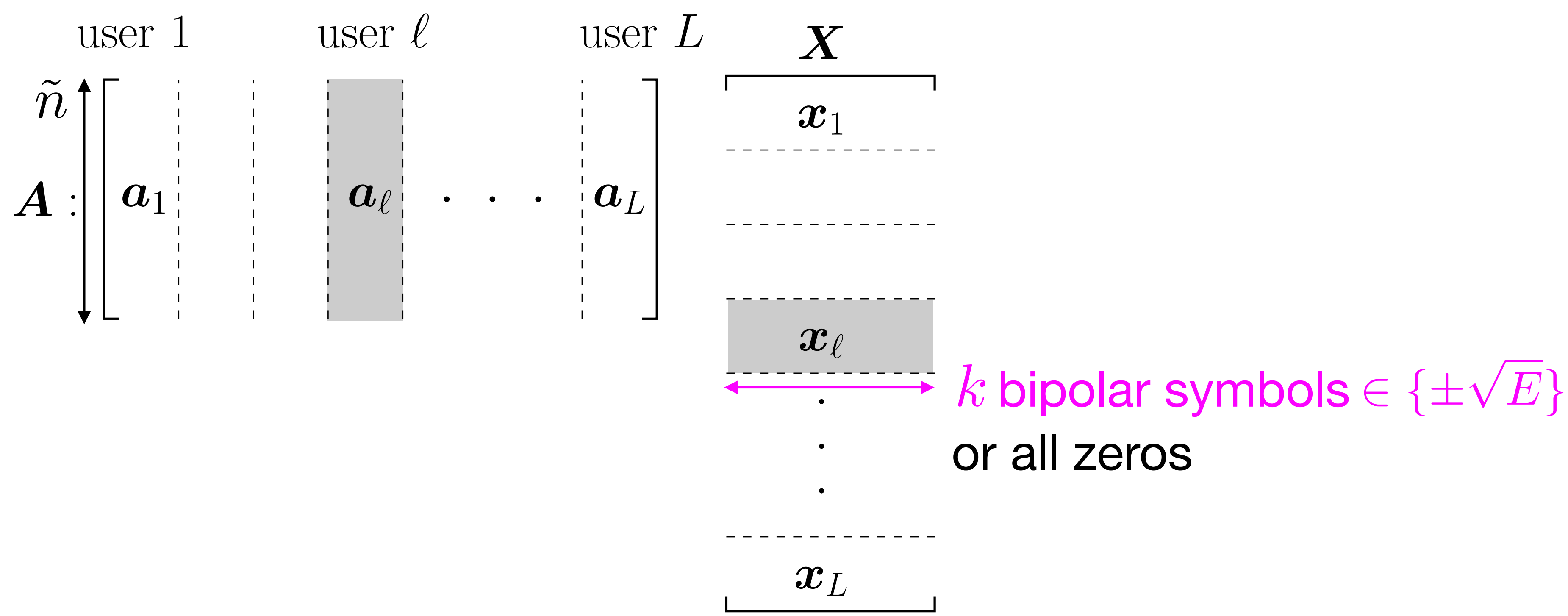
Efficient scheme

Random binary-CDMA

channel uses
$n = \tilde{n}k$



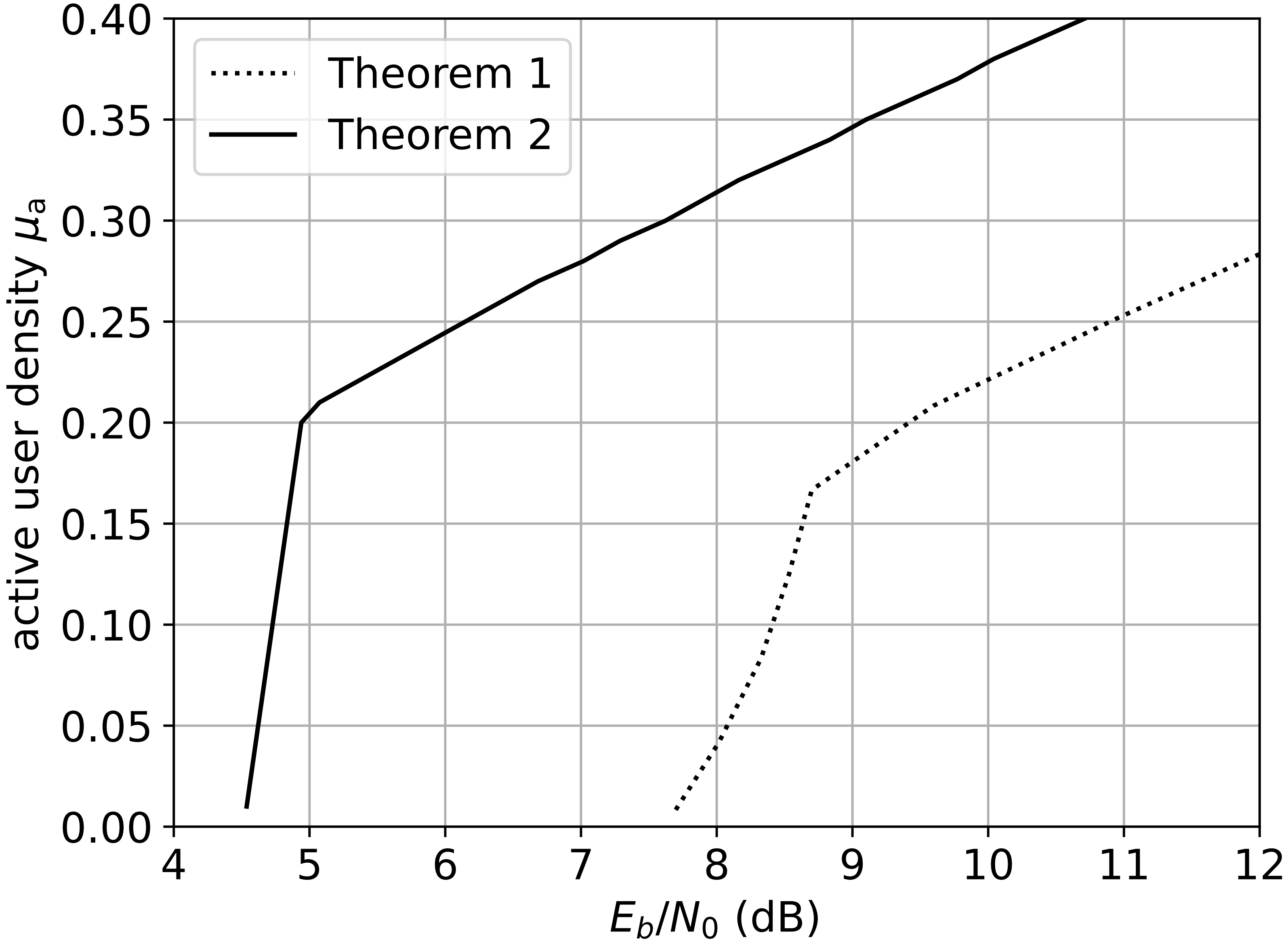
- Channel output: $\boldsymbol{Y} = \sum_{\ell \in [L]} \boldsymbol{a}_\ell \boldsymbol{x}_\ell + \boldsymbol{\mathcal{E}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{\mathcal{E}} \in \mathbb{R}^{\tilde{n} \times k}$
- $\boldsymbol{x}_\ell \in \{\pm\sqrt{E}\}^k$ with prob. α, or all-zero with prob. $1 - \alpha$

Approximate Message Passing (AMP) decoding

- AMP decoder can be tailored to prior distribution of $\boldsymbol{X}$
- Asymptotic $p_{\mathrm{MD}}, p_{\mathrm{FA}}, p_{\mathrm{AUE}}$ can be precisely quantified

$$k = 6,\ \alpha = 0.7,$$

target total error $\max\{p_{\mathrm{MD}}, p_{\mathrm{FA}}\} + p_{\mathrm{AUE}} \leq 0.01$



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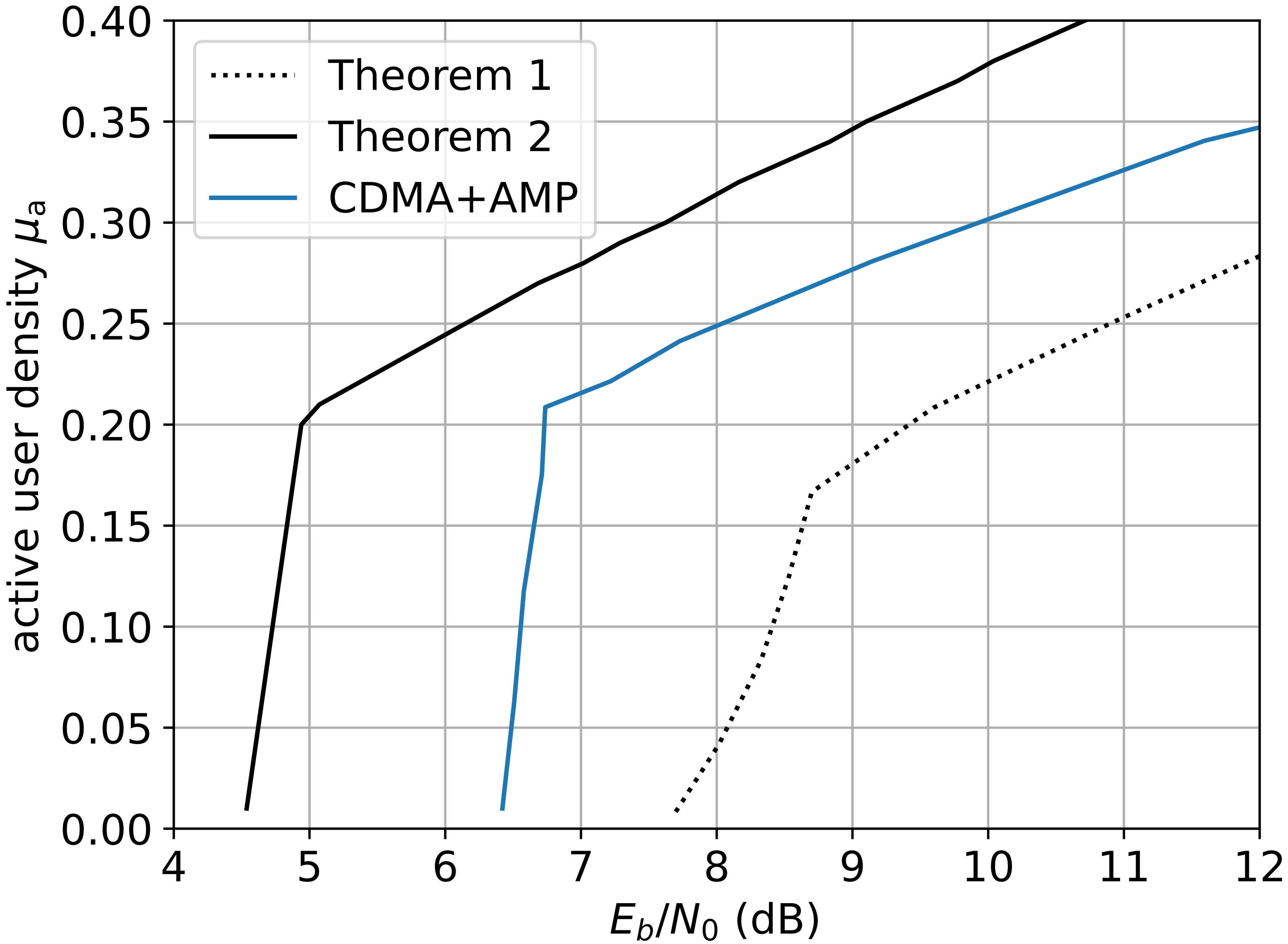

Summary

Many-user Gaussian multiple-access with random user activity

- Two sets of new achievability bounds
  - Finite-length bounds: random coding + maximum-likelihood decoding
  - Asymptotic bounds: random coding + AMP decoding
- Efficient schemes via random binary-CDMA + AMP decoding

- **Future directions**: compare with converse bounds;
  extension to unsourced random access

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